UNDERSTANDING GROUPS VIA THEIR CAYLEY GRAPHS

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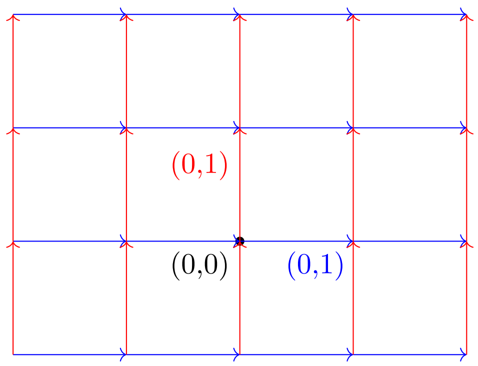
1. Geometric Group Theory and the Cayley graph

Geometric Group Theory is an approach to the study of finitely generated groups by looking at their action on certain topological spaces. The Cayley graph of a given group is such a nice space.

Chosen a (finite) set of generators $S \subset G$, it has a vertex for every element of the group and an (oriented) edge from a vertex corresponding to an element $g \in G$ to another corresponding to an element $h \in G$ if there is a generator $s \in S$ such that $sg = h$. Loops in the Cayley graphs correspond to relations between the generators. A group acts on its Cayley graph by automorphisms.



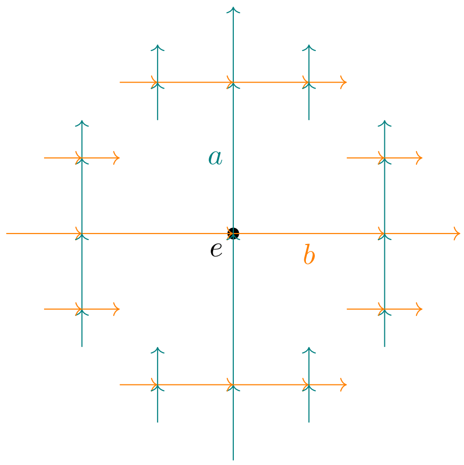


Figure 1. From left to right: the Cayley graph of $\mathbb{Z}^2$ generated by $(0,1)$ and $(1,0)$ and the Cayley graph of the free group F_2 with two generators a and b.

By importing classical notions from Graph Theory, there arise concepts of hyperbolicity, rigidity, and metrics for groups. In modern mathematics, this turns out to be a useful tool in various areas, such as hyperbolic geometry, classification, low-dimensional topology, combinatorics, optimization and many others.

2. The lamplighter group

The lamplighter group is a remarkable example for which the group and the graph structures can be understood the one though the other. Picture it thinking of a lamplighter walking along an infinite street with infinitely many lamplights, switching the state of a finite number of them and ends their journey underneath a lamplight. Such operations are the elements of the group L_2. The binary operation of L_2 is defined by concatenating such moves. This group can formally be described as

$$L_2 = \langle a,t \rangle \Big/ \{a^2 = 1; [t^k a t^{-k}, t^j a t^{-j}] = 1 \; \forall k, j \in \mathbb{Z}\} .$$

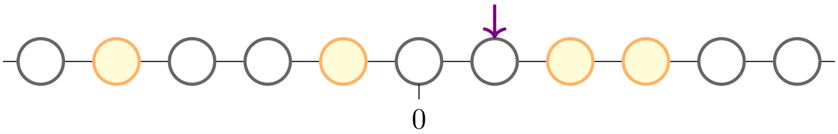


FIGURE 2. The element $[t^{-4}at^3at^3atat^{-2}] \in L_2$ represents the following move: switching the state of the lamps that are 4 and 1 position to the left of the starting point, switching the state of the lamps that are 2 and 3 positions to the right of the starting point and ending their journey 1 position to the right of the starting point.

This project will focus on the study of this group with a geometric approach. Possible directions for such investigation include:

- comparing 3 different descriptions for this group: the one above, one in terms of a semidirect product and a matrix presentation;
- understanding its Cayley graph and small neighborhoods of its vertices, and what piece of information do they give on the group itself;
- relate generalization of the Cayley graph of L_2 to generalized lamplighter groups L_k (with lamps admitting k states).

BACKGROUND: Linear Algebra, Group Theory
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REFERENCES

[1] Matt Clay, Dan Margalit, *Office Hours with a Geometric Group Theorist*, Princeton University Press, Year: 2017
[2] John Meier, *Groups, Graphs and Trees. An Introduction to the Geometry of Infinite Groups*, London Mathematical Society Student Texts 73, 2008
[3] Wolfgang Woess, *What is a horocyclic product, and how is it related to lamplighters?*, Internat. Math. Nachrichten Nr. 224 (2013), 1–27, arxiv:1401.1976