Knot theory and algebraic curves: Milnor's formula

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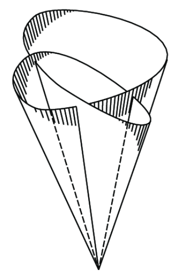

Figure 1: A trefoil knot as the link of the singularity $z^2 + w^3 = 0$

1 Introduction

What does a knot theorist know about singular algebraic curves? The focus of this project is to understand how knot theory interacts with the theory of complex curve singularities. We will work towards Milnor's celebrated formula

$$2\delta = \mu + r - 1.$$

2 Prerequisites

Basic knowledge on topology and/or algebraic curves would be very helpful, also if relevant courses are taken in parallel with the project. Depending on your background and interests we can modify our plans and goals!

3 Project Description

Going back to Newton, the study of curves singularities has been a central problem of mathematics. By curve singularities we mean objects that look like

$$f(z_1, z_2) = 0$$

for some complex polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ where $f'(0) = 0 = f(0)$. These objects have one complex dimension or, equivalently, two real dimensions and so, in principle, we can ask:

Question 1. *What is the topology of curve singularities? What topological surfaces do they represent? In other words, how can we draw them?*

We can answer this question by studying the object know as the **Milnor link** L_f of a singularity $f : \mathbb{C}^2 \to \mathbb{C}$. Roughly, the Milnor link is the intersection of a sphere, having ϵ small radius, with the singular curve, i.e.

$$L_f = S^3(\epsilon) \cap f^{-1}(0).$$

We can show that L_f is a compact, 1-dimensional topological space and so it is **just a collection of circles** sitting inside the sphere $S^3(\epsilon)$. The study of such collections is the the mathematical discipline of **knot theory**. One central focal point of this project is:

Goal 1. *Understand concretely how to associate knots to singularities.*

For example, as sketched in Figure 1, we can show that the singularity $f : z^2 + w^3 = 0$ has as link L_f the classical *trefoil knot*!

The link L_f of a singularity f creates a bridge between algebraic geometry and topology. One broad question that has motivated a lot of very beautiful and deep research of the last 100 years is the following:

Question 2. *What algebraic information of the singularity f is reflected to the topology of its link L_f?*

In his seminal book [4], Milnor sets up the general theory for links of singularities for singularities in arbitrary many variables and gives many different answers to the question above. In Chapter 10 of his book, he explains in detail one particular instance of such an interaction between algebra and topology: He considers a curve singularity f and proves the formula:

$$2\delta = \mu + r - 1. \tag{1}$$

The number δ is the *number of double points* of a simplification of f, *the algebraic information*, while μ and r are certain ubiquitous quantities related to the knot L_f, *the topological information.*

Goal 2. *Understand all the quantities appearing on Milnor's formula and sketch the proof appearing in [4].*

4 Further/alternative directions

Depending on your interest, we can focus more on the algebraic side or the knot theoretic side of the project. If there is much enthusiasm, we can look together what happens in higher dimensions: For complex *surface* singularities, their link is a compact oriented real 3-dimensional manifold therefore understanding the interplay between topology and geometry becomes more nuanced. A very nice and ambitious goal would be to examine together *Mumford's theorem*: The geometry of the singularity is trivial if and only the topology of its link is trivial!

5 Some literature

Here is some literature that you can have a look at if the contents of the project sound interesting. It's meant only to spark curiosity and it absolutely doesn't mean that you need to know all these things to start the project!

1. The main reference for this project (and for the field in general) is Milnor's beautiful book [4]. Equal in beauty but more historical, is the very thorough treaty of Brieskorn and Knorrer [2], in particular Part III.
2. Knot theory: There are many nice books on Knot theory but my favorite has to be Prasolov's book [7].
3. Relevant background material needed for algebraic curves (and much more!) can be found in Francis Kirwan's book [3]. Any book on curves will do but this is focused for curves defined over $\mathbb{C}$ so it's a bit more suited for our needs.
4. Mumford's theorem was first proven here [5].
5. For a glimpse of what current research looks like for Links of singularities (mostly in higher dimensions) you can check Chris Peter's notes [6] and our paper [1]. Both of these articles came out of a seminar organized in Leiden two years ago!

References

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