Graph series and quantum modularity

0.1. **Graph series.** To a simple graph G with r vertices $v_1, \ldots, v_r$ and set of edges E, one can associate a formal q-series by taking:

$$H_G(q) := \sum_{(n_1,\ldots n_r)\in\mathbb{Z}^r_{\geqslant 0}} \frac{q^{\sum\limits_{1\leqslant i<j\leqslant r} c_{ij}n_in_j + n_1+\cdots+n_r}}{(q)_{n_1}\ldots(q)_{n_r}},$$

where $c_{ij} = \begin{cases} 1 & v_iv_j \in E \\ 0 & v_iv_j \notin E \end{cases}$ and $(q)_m = \prod_{j=1}^m (1-q^j)$ ([2]). For example, for a graph L with 2 vertices and 1 edge we have $H_L(q) := \sum_{(n,m)\in\mathbb{Z}^r_{\geqslant 0}} \frac{q^{nm+n+\cdots+m}}{(q)_n\cdots(q)_m}$. This definition can be extended further to non-simple graphs, as well as to directed graphs. Such graph series are related to the Hilbert series of jet schemes.

0.2. **Quantum modularity.** By taking $q = e^{2\pi ix}$ with $x \in \mathbb{C}$, certain graph series as above have been shown to have *modularity* properties and even be related to objects in physics such as vertex operator algebras. More precisely, up to some fixed q-factor, the graph series of certain graphs have been shown to be *quantum modular.*

Definition 0.1 ([3]). *A* **quantum modular form** *of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is a function $f : \mathbb{Q} \to \mathbb{C}$ such that the function:*

$$h_\gamma(x) = f(x) - (cx+d)^{-k} f(\frac{ax+b}{cx+d}) : \mathbb{Q} \to \mathbb{C}$$

can be extended to a real analytic function $\widetilde{h} : U \to \mathbb{C}$, where U is an open subset of $\mathbb{R}$.

Examples of quantum modular forms of weight 1 are:

(1) the divisor sum series: $D(q) = \sum\limits_{n\geqslant 1} \sigma(n)q^n = \sum\limits_{n\geqslant 1} \frac{q^n}{1-q^n}$, where $\sigma(n) = \sum_{d|n} 1$ (see [1]).

(2) $q(q)^2_\infty H_{A_4}q$, where A_4 is the path graph $1-2-3-4$ with 4 vertices and $(q)_\infty = \prod_{n=1}^\infty (1-q^n)$ (see [2])

0.3. **New cases.**

Problem 1. *Some interesting cases to consider:*

- *graph with 2 vertices and k edges*
- *cycle graph with r vertices*

Are the graph series (up to a q-factor) quantum modular?

Problem 2. *A different question is to see if it is possible to find $\vec{b} := (b_1, \ldots, b_r)$ and $\vec{c} = (c_1, \ldots, c_r)$ in $\mathbb{Z}^r$ such that the modified graph series:*

$$H_G(q; \vec{b}, \vec{c}) := \sum_{(n_1,\ldots n_r)\in\mathbb{Z}^r_{\geqslant 0}} \frac{q^{\sum\limits_{1\leqslant i<j\leqslant r} c_{ij}n_in_j + b_1n_1+\cdots+b_rn_r}}{(q)_{n_1}\ldots(q)_{n_r}}$$

is modular. One can consider cases of graphs with $r \leqslant 4$ vertices.

References

[1] S. Bettin, J. Conrey, *Period functions and cotangent sums*, Algebra & Number Theory 7.1 (2013): 215-242.

[2] K. Bringmann, C. Jennings-Shaffer, A. Milas, *Graph schemes, graph series, and modularity*, Journal of Combinatorial Theory, Series A 197 (2023): 105749.

[3] D. Zagier, *Quantum modular forms*, Quanta of maths 11.659-675 (2010): 5