Topics in ANT. LECTURE 11.

LAST TIME: Two descriptions of space of measures

$$\mathrm{Meas}(\mathbb{Z}_p, L) \cong \Lambda(\mathbb{Z}_p) \otimes L \quad \text{(Iwasawa algebra, "measure = fn on compact opens")}$$

$$\cong \mathcal{O}_L[[T]] \otimes L \quad \text{(Mahler transform)}$$

TODAY: ① Study operations on measures

② Define Kubota-Leopoldt p-adic zeta function!

§1. Operations on measures

Suppose $\mu \in \mathrm{Meas}(\mathbb{Z}_p, L)$. Then can

a) multiply it by a function

b) restrict it to compact open (like $\mathbb{Z}_p^\times$)

c) act by $\mathbb{Z}_p^\times$, and by operators φ, ψ.

a) Suppose $f \in \mathrm{Cont}(\mathbb{Z}_p, L)$, then define $f\mu \in \mathrm{Meas}(\mathbb{Z}_p, L)$ by

$$\int_{\mathbb{Z}_p} g(x) \cdot f\mu(x) = \int_{\mathbb{Z}_p} f(x) g(x) \cdot \mu(x)$$

When $f(x) = x$: then have

$$\int_{\mathbb{Z}_p} \binom{x}{n} \cdot x\mu = \int_{\mathbb{Z}_p} (x - n + n) \binom{x}{n} \cdot \mu(x)$$

$$= \int_{\mathbb{Z}_p} (n+1) \binom{x}{n+1} \cdot \mu + \int_{\mathbb{Z}_p} n \binom{x}{n} \cdot \mu$$

so $\mathcal{A}_{x\mu}(T) = \partial \mathcal{A}_{\mu}(T)$

where

$$\partial := (1+T)\frac{d}{dT}.$$

When $f(x) = z^x$, some $z \in 1 + p\mathbb{Z}_p$

then have

$$\mathcal{A}_{z^x\mu}(T) = \mathcal{A}_\mu((1+T)z - 1)$$

$$= \int_{\mathbb{Z}_p} (1+T)^x z^x \quad \mu$$

$$= \sum_{n \geq 0} \left(\int_{\mathbb{Z}_p} z^x \binom{x}{n} \cdot \mu(x) \right) T^n$$

Very important in what follows will be the "moments"

$\int_{\mathbb{Z}_p} x^k \cdot \mu$ of a measure μ, $k \geq 0$

which by above may be computed as

$$\int_{\mathbb{Z}_p} x^k \cdot \mu = (\partial^k \mathcal{A}_\mu)(0)$$

b) Suppose $X \subset \mathbb{Z}_p$ compact open, let $\mathbb{1}_X$ be its characteristic function.

Define $\mathrm{Res}_X(\mu)$ by

$$\int_{\mathbb{Z}_p} g(x) \cdot \mathrm{Res}_X(\mu) = \int_{\mathbb{Z}_p} g(x)\, \mathbb{1}_X \cdot \mu(x)$$

When $X = a + p^n \mathbb{Z}_p$ then

$$\mathbb{1}_X(x) = \frac{1}{p^n} \sum_{\zeta^{p^n}=1} \zeta^{x-a}$$

and we use this to check that

$$\mathcal{A}_{\mathrm{Res}_X(\mu)}(T) = \frac{1}{p^n} \sum_{\zeta^{p^n}=1} \zeta^{-a} \mathcal{A}_\mu\big((1+T)\zeta - 1\big),$$

When $X = \mathbb{Z}_p^\times$ then

$$\mathcal{A}_{\mathrm{Res}_{\mathbb{Z}_p^\times}(\mu)}(T) = \mathcal{A}_\mu(T) - \frac{1}{p} \sum_{\zeta^p=1} \mathcal{A}_\mu\big((1+T)\zeta - 1\big)$$

c) Actions of $\mathbb{Z}_p^\times$, φ, ψ:

- Let $a \in \mathbb{Z}_p^\times$, then define $\sigma_a(\mu)$ by

$$\int_{\mathbb{Z}_p} g(x) \cdot \sigma_a(\mu) = \int_{\mathbb{Z}_p} g(ax) \cdot \mu$$

(defines action $\mathbb{Z}_p^\times \circlearrowright \mathrm{Meas}(\mathbb{Z}_p, L)$)

$$\mathcal{A}_{\sigma_a(\mu)}(T) = \mathcal{A}_\mu\left((1+T)^a - 1\right)$$

- Define $\varphi(\mu)$ by

$$\int_{\mathbb{Z}_p} g(x) \cdot \varphi(\mu) = \int_{\mathbb{Z}_p} g(px) \cdot \mu$$

$$\mathcal{A}_{\varphi(\mu)}(T) = \mathcal{A}_\mu\left((1+T)^p - 1\right)$$

- Define $\psi(\mu)$ by

$$\int_{\mathbb{Z}_p} g(x) \cdot \psi(\mu) = \int_{p\mathbb{Z}_p} g(p^{-1}x) \cdot \mu$$

$$\mathcal{A}_{\psi(\mu)}(T) = ??$$

The operator ψ is more mysterious ...
We know

$$\begin{cases} \psi \circ \sigma_a = \sigma_a \circ \psi \\ \varphi \circ \sigma_a = \sigma_a \circ \varphi \end{cases} \qquad \begin{cases} \psi \circ \varphi(\mu) = \mu \\ \varphi \circ \psi(\mu) = \mathrm{Res}_{p\mathbb{Z}_p}(\mu) \end{cases}$$

All we can say about $\mathcal{A}_{\psi(\mu)}(T)$ is the following:

① $$\mathcal{A}_{\psi(\mu)}\left((1+T)^p - 1\right) = \frac{1}{p} \sum_{\zeta^p = 1} \mathcal{A}_\mu\left((1+T)\zeta - 1\right)$$

② Suppose μ is such that its Mahler transform is of the shape

$$\mathcal{A}_\mu(T) = \sum b_n (1+T)^n, \quad b_n \in L.$$

then

$$\mathcal{A}_{\psi(\mu)}(T) = \sum_{n \geq 0} b_{np} (1+T)^n.$$

Proof: We know that

$$(\varphi \circ \tilde{\psi})(\mathcal{A}_\mu) = \frac{1}{p} \sum_{n \geq 0} \underbrace{\left(\sum_{\zeta^p = 1} b_n \zeta^n \right)}_{= \begin{cases} 0 & \text{if } p \nmid n \\ p\, b_n & \text{if } p \mid n. \end{cases}} (1+T)^n$$

$\varphi(\mathcal{A}_{\tilde{\psi}(\mu)}(T))$

$$= \sum_{m \geq 0} b_{mp} (1+T)^{mp}$$

But also $\varphi\left(\sum_{m \geq 0} b_{mp} (1+T)^m \right)$ (*)

$$= \sum_{m \geq 0} b_{mp} (1+T)^{mp}.$$

So the equality follows if we show φ is injective. This is clear, since $\int_{\mathbb{Z}_p} g(x) \cdot \psi(\mu) = \int_{\mathbb{Z}_p} g(px) \cdot \mu$

and $\mathrm{Cont}(\mathbb{Z}_p, L) \longrightarrow \mathrm{Cont}(\mathbb{Z}_p, L)$ is surjective.
$g(x) \longmapsto g(px)$

§2. p-Adic L-functions

Will study L-functions of $\mathbb{Q}$.

Ostrowski tells us that if we want to do analysis, we can choose to do it at any place v of $\mathbb{Q}$, and the only choices are $v = \infty$ and $v = p$ prime

$v = \infty$ The Riemann zeta function.

Lemma: Suppose $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is C^∞ with exponential decay at infinity.

Then
$$L(f, s) := \frac{1}{\Gamma(s)} \int_0^\infty f(t)\, t^s \frac{dt}{t}$$

where $\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$, $\mathrm{Re}(s) > 0$,

① has analytic continuation to all $s \in \mathbb{C}$.

② has special values
$$L(f, -n) = (-1)^n \left(\frac{d}{dt}\right)^n f(0)$$
$n \geq 0$.

For Riemann, we choose

$$f(t) = \frac{t}{e^t - 1} \in C^\infty + \text{exponential decay}$$

we find

$$L(f, s-1) = (s-1) \cdot \underbrace{\sum_{n \geq 1} n^{-s}}_{=: \zeta(s)}$$

so we find immediately that

① $\zeta(s)$ has analytic continuation to $s \in \mathbb{C}$ except simple pole at $s=1$ of residue 1.

② $\zeta(1-k) = -B_k/k$, where

$$\frac{t}{e^t - 1} = \sum B_k \, t^k/k! \quad \text{Bernoulli numbers}$$

$v = p$ The Kubota-Leopoldt zeta function

Steps
- A. "Smoothened" zeta functions
- B. Pseudo-measures
- C. Kubota-Leopoldt.

A. Want to take "Mellin transform" of measure μ_a, $a \in \mathbb{Z}_p^\times \cap \mathbb{N}$, defined by its Mahler transform

$$\mathcal{A}_{\mu_a}(T) = \frac{1}{T} - \frac{a}{(1+T)^a - 1} \in \mathbb{Z}_p[[T]]$$

If we formally substitute $T = e^t - 1$ we obtain

$$f_a(t) = \frac{1}{e^t - 1} - \frac{a}{e^{at} - 1}$$

$$= \sum_{n \geq 0} (1 - a^{n+1}) \cdot \frac{B_{n+1}}{n+1} \cdot \frac{t^n}{n!}$$

Note that $\partial = (1+T)\frac{d}{dT}$ becomes $\frac{d}{dt}$!

So we obtain the moments of μ_a from f_a as follows:

$$\int_{\mathbb{Z}_p} x^k \cdot \mu_a = (\partial^k f_{\mu_a})(0)$$
$$= \left(\frac{d}{dt}\right)^k f_a(0)$$
$$= (1-a^{k+1})\frac{B_{k+1}}{k+1}$$
$$= (-1)^k(1-a^{k+1})\,\zeta(-k)$$

Finally, its restriction to $\mathbb{Z}_p^\times$ has moments given by

$$\int_{\mathbb{Z}_p^\times} x^k \cdot \mu_a = (-1)^k(1-a^{k+1})\underbrace{(1-p^k)}_{!}\,\zeta(-k)$$

<u>Why</u>?

Proof: First, we will show that

$$\gamma_\varphi(\mu_a) = \mu_a .$$

Note that

$$A_{\mu_a}(T) = \frac{a}{1-(1+T)^a} - \frac{1}{1-(1+T)}$$

$$= \sum \lambda_n (1+T)^n$$

where

$$\lambda_n = \begin{cases} a-1 & \text{if } a \mid n \\ -1 & \text{if } a \nmid n \end{cases}$$

since $p \nmid a$, we have $\lambda_{np} = \lambda_n$
so by Lemma, we find $\gamma_\varphi(\mu_a) = \mu_a$.

Apply $\mathrm{Res}_{\mathbb{Z}_p^\times}(\mu_a) = (1 - \varphi \cdot \gamma)(\mu_a)$

$$= (1-\varphi)(\mu_a)$$

To deduce that

$$\int_{\mathbb{Z}_p^\times} x^k \cdot \mu_a = \int_{\mathbb{Z}_p} x^k \cdot (1-\varphi)\mu_a$$

$$= (1-p^k) \int_{\mathbb{Z}_p} x^k \cdot \mu_a \qquad \square.$$

<u>Next time</u>: Want to get rid of dependence on a!

$$\int_{\mathbb{Z}_p^\times} x^k \cdot \mu_a = \pm \underbrace{(1-a^{k+1})}_{\text{Yuck!}}(1-p^k)\zeta(-k)$$

Note that

$$(1-a^{k+1}) = \int_{\mathbb{Z}_p^\times} x^k \cdot \, x([1]-[a])$$

Dirac measures at 1 and a.

<u>WANT</u>: To divide by $x([1]-[a])$.