# MAHLER MEASURE

ROBIN DE JONG, EVGENY VERBITSKIY

Consider a polynomial $P$ in several variables with integer coefficients, i.e.,

$$P(z_1,\dots,z_d) = \sum_{k_1,\dots k_d\ge 0} c_{k_1,\dots,k_d} z^{k_1}\dots z_d^{k_d},$$

where $c_k = c_{k_1,\dots,k_d} \in \mathbb{Z}$, and there are only finitely many $k$'s such that $c_k \neq 0$.

The (logarithmic) Mahler measure is given by

$$m(P) = \int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_d})|\, d\theta_1\cdots d\theta_d.$$

It turns out that such integrals are well defined, even if $P$ has roots in $\mathbb{S}^d = \{z = (z_1,\dots,z_d):\ |z_1| = \ldots = |z_d| = 1\}$. Moreover, $m(P) \ge 0$ and $m(PQ) = m(P) + m(Q)$ for all polynomials $P$,$Q$ with integer coefficients.

If $P$ is a polynomial in one variable, $P(z) = c_0 + c_1 z + \ldots + c_d z^d$, $c_0 \neq 0$ with roots $\alpha_1,\dots,\alpha_d \in \mathbb{C}$, then the logarithmic Mahler measure of $P$ is given by

$$m(P) = \log|c_0| + \sum_{j:|\alpha_j|>1} \log|\alpha_j|.$$

The Mahler measure is **ubiquitous** in mathematics: it appears naturally and frequently in algebraic number theory, analysis, dynamical systems, and even statistical physics and electrical engineering.

The famous (unsolved) Lehmer's conjecture states that for any non-zero polynomial with integer coefficients $P$, either $m(P) = 0$ or $m(P) \ge c > 0$.

The goal of this project is to get familiar with the basic propertiesand fundamental results about the Mahler measure. The possible research question is related to the interplay between the Mahler measure and the height of a polynomial $h(P) = \max|c_k|$. More specifically, if $P$ is a polynomial in $d$ variables with $m(P) > 0$, then for all sufficiently large $N$, there exists no non-trivial polynomial $Q = \sum_k c_k z_1^{k_1 N}\dots z_d^{k_d N}$ with $|c_k| \le 2$ for all $k \in \mathbb{Z}_+^d$ such that $Q$ is divisible by $P$.

## References

[1] F. Brunault and W. Zudilin, *Many Variations of Mahler Measures: A Lasting Symphony*, Australian Mathematical Society Lecture Series, Cambridge University Press, Cambridge, 2020.