ELLIPSES, THE NUMERICAL RANGE, AND PONCELET'S PORISM

FRANCESCA ARICI

This project concerns a surprising connection between complex analysis, linear analysis, and projective geometry.

Given an $n \times n$ matrix A, recall that the *numerical range* $W(A)$ is defined by

$$W(A) := \{\langle Ax, x\rangle : \; x \in \mathbb{C}^n, \; ||x|| = 1\}.$$

This set contains the spectrum of A, it is a convex set, and its outer boundary is a convex curve.

The *elliptical range theorem* states that if A is a 2×2 matrix, then its numerical range is either a point, a line segment, or an elliptical disk. Although this is a theorem about 2×2 matrices, it also sheds light on the numerical range of $n \times n$ matrices.

A finite *Blaschke product* B of degree n is a function of the form

$$B(z) = \gamma \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z},$$

where $a_j \in \mathbb{D}$ for $j = 1, \dots, n$ and $\gamma \in \mathbb{T}$.

Note that Blaschke products of degree 1 are the disk automorphisms. Finite Blaschke products are n to 1 maps of the open unit disk $\mathbb{D}$ into itself and the unit circle $\mathbb{T}$ to itself. They are holomorphic on an open set containing the closed unit disk and have finitely many zeros in $\mathbb{D}$.

We will consider the set of points in $\mathbb{T}$ that a Blaschke product identifies; in other words, we will be interested in the solutions of $B(z) = \lambda$ for $\lambda \in \mathbb{T}$. We will see that, quite naturally, an ellipse appears when looking at those points. We will refer to this as the Blaschke ellipse or the ellipse associated to the Blaschke product.

In order to understand why ellipses appear, we will focus on two classical results from projective geometry: Poncelet porism and the Darboux theorem.

Theorem 0.1 (Poncelet)**.** *If there exists a polygon of n-sides that is inscribed in a given conic and circumscribed about another conic, then infinitely many such polygons exist.*

This theorem is sometimes called Poncelet's porism and the related polygons were called Poncelet's polygons. The two conics are said to be *n-Poncelet related.*

Moreover, the following is true for ellipses inscribed in the unit circle:

Theorem 0.2 (Darboux)**.** *Let E_1 be an ellipse inscribed in a convex n-gon that is, in turn, inscribed in $\mathbb{T}$. Consider the diagonals of the Poncelet polygons that leap over m vertices (i.e., all such diagonals that connect vertex $k-1$ with vertex $k+m$ where $k = 1, 2, \dots, n$ and the vertex numbers are taken modulo n). The envelope of these diagonals is also an ellipse E_{m+1} for $m = 1, \dots, [n/2] - 1$.*

The goal of this project is to study the following result by [1], and to relate it to these classical theorem from projective geometry.

Theorem 0.3. *An ellipse E is inscribed in a quadrilateral that is itself inscribed in the unit circle if and only if E is the ellipse associated with a Blaschke product $B = C \circ D$, where C and D are degree-2 Blaschke products.*

Prerequisites: linear algebra, linear analysis, complex analysis.

References

[1] M. Fujimura, Inscribed ellipses and Blaschke products. Comput. Methods Funct. Theory 13 (2013), no. 4, 557–573.

Email address: `f.arici@math.leidenuniv.nl`