Translation surfaces from polygons: topology, geometry and dynamics

Bachelor project

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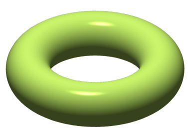

Figure 1: Torus

Take a unit square $[0,1] \times [0,1]$ and identify the opposite edges. The resulting surface is a *torus*, i.e. the surface of a doughnut. The torus is a compact surface of genus one of unit area. From the point of view of dynamics, one can study the *parallel flow* on the torus: imagine a particle, moving on the surface of the torus, from the bottom to the top of the square at an angle α. Then the *trajectory* of the particle is a straight line of slope $\tan\alpha$. When the particle reaches, say, the top edge of a square, via the identification of edges it is transferred to the bottom edge, and continues its motion along a straight line. For a given α, either every trajectory is a circle, or it is a line winding densely around the torus. The dynamics of the trajectory intersecting a horizontal section can be described using an *interval exchange transformation*.

If we subdivide the edges of the square into subintervals (possibly an infinite number of them), and then identify them by translations, we may obtain surfaces with more complicated topology: of genus 2, 3, or even infinite genus. In order to make identifications in a consistent way, we often must remove points (singularities) from the square, which produces *punctures* in the resulting surface. A punctured surfaces is non-compact, and all such surfaces are distinguished by their genus and the number of ends, the latter corresponding to the number of punctures. The dynamics of the parallel flow on such a surface becomes a lot more complicated, but it still can be described by an interval exchange transformation. Such dynamical systems are actively studied in the modern mathematics. The case when the resulting surface is of infinite genus is of most interest, as very little is known about the dynamics of the parallel flow in this case.

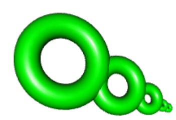

Figure 2: Surface of infinite genus and finite area with one end

During the project, the student will learn theory of classification for non-compact surfaces by genus and the number of ends, and the basic methods to study the dynamics for parallel flows on translation surfaces. Depending on the background and preferences of the student, an emphasis may be made towards either of these topics. Possible directions include: translation surfaces of finite or infinite types, algebraic objects associated to finite translation surfaces (i.e. Veech groups, Teichmüller spaces), methods to study the dynamics of infinite translation surfaces (Bratteli diagrams), etc.

References

[1] H. Bruin and O. Lukina, *Rotated odometers and actions on rooted trees*, Fund. Math., available online at https://www.impan.pl/en/publishing-house/journals-and-series/fundamenta-mathematicae/online/114917/rotated-odometers-and-actions-on-rooted-trees.

[2] V. Delecroix, P. Hubert, F. Valdez, *Infinite translation in the wild*, Chapters 1 and 2, available at https://www.labri.fr/perso/vdelecro/infinite-translation-surfaces-in-the-wild.html.

[3] M. Keane, *Interval exchange transformations*, Math. Z., 141 (1975), 25–31.

[4] I. Richards, *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc., 106 (1963), 259–269.