MAHLER MEASURE

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Consider a polynomial P in several variables with integer coefficients, i.e.,

$$P(z_1,\ldots,z_d)=\sum_{k_1,\ldots k_d\geq 0} c_{k_1,\ldots,k_d} z^{k_1}\ldots z_d^{k_d},$$

where $c_k = c_{k_1,\ldots,k_d}\in\mathbb{Z}$, and there are only finitely many k's such that $c_k\neq 0$.

The (logarithmic) Mahler measure is given by

$$m(P)=\int_0^1\cdots\int_0^1 \log|P(e^{2\pi i\theta_1},\ldots,e^{2\pi i\theta_d})|\,d\theta_1\cdots d\theta_d.$$

It turns out that such integrals are well defined, even if P has roots in $\mathbb{S}^d=\{z=(z_1,\ldots,z_d):\ |z_1|=\ldots=|z_d|=1\}$. Moreover, $m(P)\geq 0$ and $m(PQ)=m(P)+m(Q)$ for all polynomials P,Q with integer coefficients.

If P is a polynomial in one variable, $P(z)=c_0+c_1z+\ldots+c_dz^d$, $c_0\neq 0$ with roots $\alpha_1,\ldots,\alpha_d\in\mathbb{C}$, then the logarithmic Mahler measure of P is given by

$$m(P)=\log|c_0|+\sum_{j:|\alpha_j|>1}\log|\alpha_j|.$$

The Mahler measure is **ubiquitous** in mathematics: it appears naturally and frequently in algebraic number theory, analysis, dynamical systems, and even statistical physics and electrical engineering.

The famous (unsolved) Lehmer's conjecture states that for any non-zero polynomial with integer coefficients P, either $m(P)=0$ or $m(P)\geq c>0$.

The goal of this project is to get familiar with the basic properties and fundamental results about the Mahler measure. The recent book by Brunault and Zudilin [1] will be a starting point for the project.

There a number of possible research questions which could be investigated. For example, various questions in relation to Dynamical Systems and Statistical Physics. Another question, arose in [2] about the interplay between the Mahler measure and the height of a polynomial $h(P)=\max|c_k|$. More specifically, if P is a polynomial in d variables with $m(P)>0$, then for all sufficiently large N, there exists no non-trivial polynomial $Q=\sum_k c_k z_1^{k_1N}\ldots z_d^{k_dN}$ with $|c_k|\leq 2$ for all $k\in\mathbb{Z}_+^d$ such that Q is divisible by P.

References

[1] F. Brunault and W. Zudilin, *Many Variations of Mahler Measures: A Lasting Symphony*, Australian Mathematical Society Lecture Series, Cambridge University Press, Cambridge, 2020.

[2] Aihua Fan, Klaus Schmidt, and Evgeny Verbitskiy, *Bohr chaoticity of principal algebraic actions and Riesz product measures*, arXiv e-prints (2021), arXiv:2103.04767.