

QUADRATIC ALGEBRAS AND THEIR HILBERT SERIES

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This project will concern the study of the growth of quadratic algebras, using the Hilbert series.

A graded algebra is called *one-generated* if the natural map $p : T(A_1) \rightarrow A$ from the tensor algebra generated by A_1 is surjective. We call a one-generated algebra *quadratic* if $J := \ker(p)$ is generated, as a two-sided ideal, by

$$I_A := J_A \cap \mathbb{T}^2(A_1) \subseteq A_1 \otimes A_1.$$

In other words, a quadratic algebra is determined by a vector space of generators $V = A_1$ and a subset $I \subset A_1 \otimes A_1$. We denote this as $A = \{V, I\}$.

For a graded vector space $V = \bigoplus_{k \geq 0} V_k$, with finite-dimensional graded components, its Hilbert series is the formal power series

$$(1) \quad h_V(z) = \sum_{k \geq 0} \dim(V_k) z^k.$$

Hilbert series of quadratic algebras (generic or not) are well-studied and understood, at least in low dimensions (cf. [2, Section 6.5]). Understanding the Hilbert series of quadratic algebras provides insight into their growth and complexity. For example, it can reveal whether the algebra is finite-dimensional or infinite-dimensional, and how the relations between generators impact the algebra's structure.

One possible direction for this project would be the study and description of Hilbert series of quadratic algebras in various dimension and numbers of relations.

This is related to the notion of a generic quadratic algebra, which in turns connect to the Golod–Shafarevich inequality, a result in algebra that relates to the growth of algebras and their representations.

Definition 1 (cf. [2, Chapter 6]). A complex *generic* quadratic algebra in d generators and r relations is a generic point in the variety $\mathcal{Q}_{d,r}$ of quadratic algebras with $\dim A_1 = d$ and $\dim(I) = r$.

Since a quadratic algebra in r relations is determined by an $(d^2 - r)$ -dimensional subspace of $\mathbb{C}^{d^2}$ (see for instance [1, Lemma 4.1]), we can identify the variety $\mathcal{Q}_{d,r}$ with the complex Grassmannian $\mathbf{Gr}_{\mathbb{C}}(r, d^2)$. A *generic* complex quadratic algebra will then mean a generic point in that Grassmannian.

In [1], Anick studied Hilbert series of associative graded algebras satisfying certain finiteness hypotheses by endowing them with a well-ordering. This allowed him to characterise generic algebras as algebras that have the generic (i.e., coefficient-wise minimal) Hilbert series.

Theorem 2 ([1, Definition 4.9]). *Let $\mathcal{L}$ be the class of Hilbert series for a class $\mathcal{G}$ of graded objects. The series $L(z) = \inf(\mathcal{L})$ is the generic series for the class $\mathcal{G}$. An object $M \in \mathcal{G}$ is generic if and only if its Hilbert series equals $L(z)$.*

In the case of quadratic algebras, one can explicitly provide the generic Hilbert series under some constraints on the number of generators and relations.

Proposition 3 (cf. [2, Proposition 4.2]). *Let A be a generic quadratic algebra in d generators and r relations satisfying*

$$r \geq \frac{3d^2}{4}, \quad r \leq \frac{d^2}{4}.$$

Then the Hilbert series of A is either

$$(2) \quad h_A(z) = (1 + dz + (d^2 - r)z^2), \quad \text{or}$$

$$(3) \quad h_A(z) = (1 - dz + rz^2)^{-1},$$

respectively.

Note that in the first case, that is, when $r \geq 3d^2/4$, (2) implies that $\dim(A_3) = 0$, hence the quadratic algebra is finite-dimensional.

An interesting question is understanding what happens in the interval that is excluded by the above proposition, where the behaviour of the Hilbert series is not fully understood.

REFERENCES

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