

A STUDY OF DYNAMICAL SYSTEMS: TOEPLITZ SUBSHIFTS

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General Background: The Banach-Tarski theorem can be stated as “given a ball in $\mathbb{R}^3$, there exists a decomposition of the ball into finitely many disjoint pieces such that these pieces can be rearranged to conform two copies of the original ball”. In other words, the Banach-Tarski theorem says that the group of rotations of $\mathbb{R}^3$ has a *paradoxical decomposition*. Despite this notion seeming to be purely algebraic, it is related to the ergodic behavior of the dynamical systems of the group.

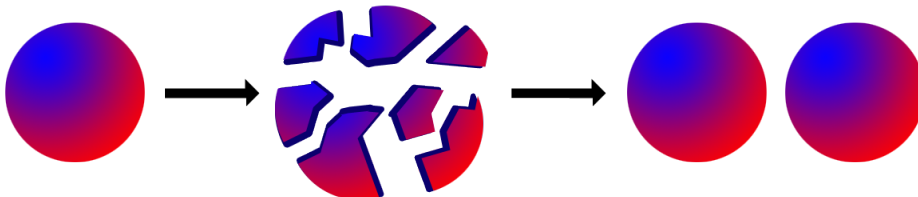


FIGURE 1. Banach-Tarski Theorem

A group which does not admit a paradoxical decomposition is called *amenable*. We can find several different characterizations of the amenability property of a group in the literature, and among all of these characterizations, we are interested in the following one (see for example [CC]): A group G is amenable if and only if every topological dynamical system on G admits at least one invariant measure.

A *topological dynamical system* for a group G is a continuous group action from G to some compact metric space X . An example of a dynamical system is the full shift: for this example, let Σ be a finite set, $G = \mathbb{Z}$, and $\Sigma^{\mathbb{Z}}$ be the set of all the possible maps $\mathbb{Z} \rightarrow \Sigma$. This set can be thought of as the set of all the possible ways to “paint” the group $\mathbb{Z}$ with Σ “colors” (see figure 2 for an example). The set $\Sigma^{\mathbb{Z}}$ is compact and metric when endowed with the product topology. Furthermore, the shift action σ from $\mathbb{Z}$ over $\Sigma^{\mathbb{Z}}$ is defined by $(\sigma^m f)(n) := f(n - m)$ for $f \in \Sigma^{\mathbb{Z}}$, $m, n, \in \mathbb{Z}$. An *invariant measure* for a dynamical system X with group G is a way (with some properties) to assign a number to every well-behaved subset of X in such manner that the action of X does not change this assignment. To gain some intuition behind this notion, consider the set $[0, 1]$ and the assignment of length to every interval contained within $[0, 1]$. Notice that translating an interval does not change its length.

Some examples of countable amenable groups are the abelian groups, for instance, the set of integer numbers with the usual sum $\mathbb{Z}$, which is the starting point to

understand the concept of dynamical systems. However, amenable groups seem to be the more adequate groups to work with due to their good properties, to study this family of groups is not enough for comprehending other groups that are of interest to mathematicians. For instance, the finitely generated free groups with at least two generators and the modular group $SL(2, \mathbb{Z})$ are not included in this family. For non-amenable groups, it is not an easy task to “count” measures on a dynamical system. For this reason, it is valuable to have necessary conditions for the existence of these measures and for counting them.

Among all the different dynamical systems we can study, we are particularly interested in the *Toeplitz subshifts*, which have played an important role for this theory.

The Toeplitz subshifts can be regarded as those subsystems in a full shift Σ^G that are generated by “almost periodic” elements. These subshifts have a deep relationship with another dynamical system known as the *G-odometer*, which can be seen as a projective limit of a certain sequence of cosets of G and provides useful information about the Toeplitz subshifts.

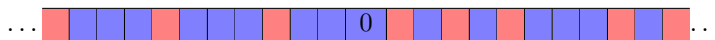


FIGURE 2. An example of a Toeplitz element for $\Sigma = \{\text{blue square}, \text{red square}\}$ and $G = \mathbb{Z}$. Each coordinate has a period a power of 2, it means that if we pick a position in $\mathbb{Z}$, there exists $n \in \mathbb{N}$ (that depends on the position) such that moving $m \in \mathbb{Z}$ times 2^n from this coordinate we find the same color. In this example, for the 0 coordinate in $\mathbb{Z}$ we use $n = 1$, for the position 1 we use $n = 2$.

Different possible tasks for the project:

- The main goal of this project is to become familiar with the notions of dynamical system, subshift, Choquet simplex, amenable groups and especially, that of Toeplitz subshift, and G -odometers.
- A problem of intermediate level is to explore different examples of dynamical systems that express the diversity regarding the existence of invariant measures for different amenable/non-amenable groups.
- A high level problem is related to the question of “counting” invariant measures of a dynamical system over a non-amenable group. We would like to find one strategy for “counting” measures in a Toeplitz subshift on non-amenable groups using the approach presented in [HLSY] and [Yoo].

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