$\mathcal{D}$-Modules, summer 2016

Elena Lavanda and Wouter Zomervrucht
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About

You are reading the program for the arithmetic geometry student seminar at FU Berlin during the summer semester of 2016. Below we sketch an outline of the content we aim to cover. Next follow some organizational remarks and a detailed schedule. At the end of the program you'll find a list of references. If you have any questions regarding the seminar or want to volunteer for a talk, please contact the organizers.

Overview

The topic of this seminar is, obviously, $\mathcal{D}$-modules. These can be seen as the algebraic version of partial differential equations. There is a striking dichotomy between the characteristic 0 and the characteristic p settings. In complex geometry, the Riemann–Hilbert correspondence is a dictionary between $\mathcal{D}$-modules and local systems, or representations of the fundamental group. In particular, all $\mathcal{D}$-modules on a proper smooth complex variety X are trivial if the fundamental group of X is trivial. We will prove and explain these statements. Then, still in characteristic 0, we make a little detour into derived categories and study some functoriality for $\mathcal{D}$-modules, as well as the Gauss–Manin connection. At the end of the seminar we move to characteristic p. An analogue of the Riemann–Hilbert correspondence was conjectured by Gieseker and recently proven by Esnault and Mehta. That proof is far beyond our scope. Instead we will study various examples of the phenomenon.

Organization

Time	Monday, 10:15–11:45
	April 18 until July 18
Location	FU Berlin, A3/SR210
Organizers	Elena Lavanda, `elenalavanda@zedat.fu-berlin.de`
	Wouter Zomervrucht, `zomervruchtw@mi.fu-berlin.de`
Website	`http://www.mi.fu-berlin.de/users/zomervruchtw/2016-dmod/`

Unfortunately, there does not seem to exist a single main reference suitable for this seminar. Therefore the schedule below includes detailed references for each talk, as well as some extra explanation where needed. However, you are free to follow any source you like; there is usually not a best one. It is only important to cover the described material. Try to include examples and counterexamples in your talk every now and then. Some have already been written down in the schedule, but the more, the merrier!

At the end of this program there is a list of references. The list is duplicated on the seminar's website with links to the full text as far as freely online available. If nevertheless you experience trouble finding a reference, just contact the organizers.

Schedule

Throughout, k is an algebraically closed field. A variety over k is an integral separated finite type k-scheme. Usually we only consider smooth varieties.

22-04. Introduction

Wouter

Give an overview of the seminar following the program below. In the second half of the talk recall the basic notions of (quasi-)coherent sheaves, derivations, differentials, and smoothness. A possible reference is [16], sections 4.2–3, 5.1, and 6.1–2.

25-04. The sheaf of differential operators

Marco

Start by introducing the sheaf of differential operators $\mathcal{D}_{X/k}$ on a smooth variety X/k. It is defined as follows. First set $\mathcal{D}^{\leq 0} = \mathcal{O}_X$. We interpret $f \in \mathcal{D}^{\leq 0}$ as the multiplication map $\mathcal{O}_X \to \mathcal{O}_X, g \mapsto fg$. Then for $i \geq 1$ define $\mathcal{D}^{\leq i}$ inductively as the subsheaf of $\mathcal{E}nd_k\, \mathcal{O}_X$ consisting of those θ for which the commutator $[\theta, f]$ with any $f \in \mathcal{D}^{\leq 0}$ lies in $\mathcal{D}^{\leq i-1}$. We obtain a system $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1} \subseteq \ldots \subseteq \mathcal{E}nd_k\, \mathrm{O}_X$ and define $\mathcal{D}_{X/k} = \operatorname{colim}_{i\geq 0} \mathcal{D}^{\leq i}$. (This colimit is just a union.) The sheaf $\mathcal{D}_{X/k}$ has a non-commutative ring structure by composition. Remark that each $\mathcal{D}^{\leq i}$ is an $\mathcal{O}_X$-module and in particular $\mathcal{D}^{\leq 1} = \mathcal{O}_X \oplus \mathcal{T}_{X/k}$ as $\mathcal{O}_X$-modules; here we identify as usual $\mathcal{T}_{X/k} = \mathcal{D}er_k(\mathcal{O}_X, \mathcal{O}_X)$. For $\eta \in \mathcal{D}^{\leq i}, \theta \in \mathcal{D}^{\leq j}$ one has $\eta\theta \in \mathcal{D}^{\leq i+j}$ and $[\eta, \theta] \in \mathcal{D}^{\leq i+j-1}$.

Since X/k is smooth, at every point of X we can find a system of local coordinates, i.e. local sections $x_1, \ldots, x_n \in \mathcal{O}_X$ such that $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ form a basis of $\Omega^1_{X/k}$. Given local coordinates, describe $\mathcal{D}_{X/k}$ as in proposition 1.8 of [14]. Make the second and third remark of corollary 1.9. Our next goal is to see this description is correct. Here is the strategy. Let $\mathcal{R}$ be the sheaf of rings given by generators and relations in proposition 1.8. There is an obvious ring homomorphism $\mathcal{R} \to \mathcal{D}_{X/k}$. Both $\mathcal{R}$ and $\mathcal{D}_{X/k}$ are filtered; by dimension considerations it suffices to prove that $\operatorname{gr} \mathcal{R} \to \operatorname{gr} \mathcal{D}_{X/k}$ is an isomorphism. For $\operatorname{char} k = 0$ that verification can be found in the proofs of propositions 5.2–3 of [8]. For $\operatorname{char} k > 0$ the same proof works.

If time permits you can now explain the fourth remark of [14], corollary 1.9. In any case mention the following distinction between characteristic 0 and p. The commutator bracket $[\cdot, \cdot]$ makes the tangent bundle $\mathcal{T}_{X/k}$ into a Lie algebra. As such, $\mathcal{T}_{X/k}$ has a universal enveloping algebra. In characteristic zero this is precisely $\mathcal{D}_{X/k}$. In positive characteristic that is false.

02-05. $\mathcal{D}$-modules and connections

Yun

Let X/k be a smooth variety. Define a $\mathcal{D}$-module on X as a sheaf of $\mathcal{D}_{X/k}$-modules, that is, a sheaf of abelian groups $\mathcal{E}$ on X endowed with (i) a map $\mathcal{D}_{X/k} \times \mathcal{E} \to \mathcal{E}$ satisfying the usual module axioms, or equivalently (ii) a ring homomorphism $\mathcal{D}_{X/k} \to \mathcal{E}nd\, \mathcal{E}$. Such $\mathcal{E}$ inherits an $\mathcal{O}_X$-module structure via $\mathcal{O}_X \to \mathcal{D}_{X/k}$ and in particular has a k-module structure. Now the map in (i) factors canonically over $\mathcal{D}_{X/k} \otimes_{\mathcal{O}_X} \mathcal{E}$. Observe also that it is k-linear but not necessarily $\mathcal{O}_X$-linear in $\mathcal{E}$, because k is the center of $\mathcal{D}_{X/k}$. Similarly the map in (ii) has image in $\mathcal{E}nd_k\, \mathcal{E}$, not necessarily in $\mathcal{E}nd_{\mathcal{O}_X}\, \mathcal{E}$. Nevertheless the map in (ii) is $\mathcal{O}_X$-linear.

If $\mathcal{E}$ and $\mathcal{F}$ are $\mathcal{D}$-modules on X, a $\mathcal{D}$-module homomorphism $\mathcal{E} \to \mathcal{F}$ is a homomorphism of abelian sheaves that respects the $\mathcal{D}$-module structure, i.e. for which the obvious diagram

$$\begin{array}{ccc} \mathcal{D}_{X/k} \times \mathcal{E} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{D}_{X/k} \times \mathcal{F} & \longrightarrow & \mathcal{F} \end{array}$$

commutes.

A first example of a $\mathcal{D}$-module is $\mathcal{D}_{X/k}$ itself. More interesting is the $\mathcal{D}$-module $\mathcal{O}_X$ with action given by $\theta \cdot f = \theta(f)$. A $\mathcal{D}$-module is called trivial if it is isomorphic as $\mathcal{D}$-module to a direct sum $\bigoplus_I \mathcal{O}_X$. To see some real examples, explain the relation between $\mathcal{D}$-modules and partial differential equations following §6.1 in [3]. In this text K denotes a field of characteristic zero and the Weyl algebra A_n is by definition the $K[x_1, \ldots, x_n]$-algebra $\mathcal{D}_{\mathbb{A}^n/K}(\mathbb{A}^n)$.

Define a stratified bundle to be a $\mathcal{D}_{X/k}$-module that is coherent as $\mathcal{O}_X$-module. Show that stratified bundles are in fact locally free: proposition VI.1.7 in [2] or proposition 1.11 in [14]. This justifies the word 'bundle'. Usually we'll restrict ourselves to stratified bundles.

Now introduce flat (or integrable) connections. Cover from [17] definition 1.1, example 1.2, example 1.5, and all of §1.3. Take $S = \operatorname{Spec} k$ and do not forget the remark at the top of page 3. Then explain how a stratified bundle induces an flat connection. Indeed, for a stratified bundle $\mathcal{E}$ the inclusion $\mathcal{T}_{X/k} \to \mathcal{D}_{X/k}$ gives a map $\mathcal{T}_{X/k} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E}$. Since $\mathcal{T}_{X/k}$ is the $\mathcal{O}_X$-dual of $\Omega^1_{X/k}$ there is an induced map $\mathcal{E} \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{E}$. Verify that this is an flat connection.

In characteristic zero one can reverse the process as follows. A flat connection on $\mathcal{E}$ induces a Lie algebra homomorphism $\mathcal{T}_{X/k} \to \mathcal{E}nd_k\, \mathcal{E}$. Recall from the previous talk that $\mathcal{D}_{X/k}$ is the universal enveloping algebra of $\mathcal{T}_{X/k}$, so $\mathcal{T}_{X/k} \to \mathcal{E}nd_k\, \mathcal{E}$ extends uniquely to a k-algebra homomorphism $\mathcal{D}_{X/k} \to \mathcal{E}nd_k\, \mathcal{E}$. Conclude that there is an equivalence of categories between stratified bundles and flat connections. This is false in positive characteristic, as then $\mathcal{D}_{X/k}$ is not the universal enveloping algebra of $\mathcal{T}_{X/k}$.

09-05. The tannakian category of $\mathcal{D}$-modules **Marcin**

Present the theory of neutral tannakian categories. An excellent reference is [5], §1–2. Do not spend too much time on all details; often saying 'the obvious diagrams commute' is enough. Most proofs can be omitted, as well as items 1.25–27, 2.12–16, 2.22–29, and 2.34–35. You may omit more where necessary, but don't skip all examples! Try to give at least a sketch of the proof of theorem 2.11.

To finish your talk, let X/k be a smooth variety and $x \in X$ a closed point. Prove that the category of stratified bundles on X is neutral tannakian with fiber functor $\mathcal{E} \mapsto \mathcal{E}_x$. Use for instance proposition 1.20 of [5]. See exercises 1.10(1) and 1.11(1) in [18] for some constructions.

16-05. No seminar

23-05. The Riemann–Hilbert correspondence **Pedro**

In complex geometry, the Riemann–Hilbert correspondence is a dictionary between $\mathcal{D}$-modules and so-called local systems. Surprisingly the latter are purely topological! Remember that, as we work in characteristic zero, stratified bundles can be replaced by flat connections.

Give a short introduction to analytic spaces. We are mostly interested in smooth analytic spaces, which are just complex manifolds. You can use sections 1.1–2 of [19] or any textbook on complex geometry. Define holomorphic flat connections as in [4], I.2.4–14. Following Deligne, give definition I.1.1 and prove proposition I.2.16 and theorem I.2.17. (The proofs are in I.2.23.) This is the classical, analytic Riemann–Hilbert correspondence. Add the remark that the equivalence is one of tensor categories, i.e. respects the tannakian structure.

Now let $X/\mathbb{C}$ be a smooth variety. Sketch how to associate to X an analytic space X^{an} and to an $\mathcal{O}_X$-module $\mathcal{F}$ an $\mathcal{O}_{X^{\mathrm{an}}}$-module $\mathcal{F}^{\mathrm{an}}$ following [19], sections 2.4–3.9. Observe in 3.9 that there is a morphism of ringed spaces $\psi\colon (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then Serre's sheaf $\mathcal{F}'$ is simply $\psi^{-1}\mathcal{F}$ and $\mathcal{F}^{\mathrm{an}} = \psi^*\mathcal{F}$.

State theorems 2 and 3 from §3.12 of [19] for proper varieties. Returning to connections, show that an algebraic (flat) connection on X induces a holomorphic (flat) connection on X^{an}. Deduce from the GAGA theorems that this construction is an equivalence of categories if X is proper. Again the equivalence is one of tensor categories. Remark that the equivalence breaks down if X is not proper; in that case one should restrict to connections with at most 'regular singularities' on a compactifaction. This is proved in [4].

30-05. Representations of the fundamental group **Fei**

Recall the Riemann–Hilbert correspondence between flat connections and local systems. On proper varieties, algebraic and analytic flat connections are the same. In this talk we'll take a closer look at local systems and try to make those algebraic as well.

First we give an second analytic characterization of local systems: on a (decent) topological space X, local systems are equivalent to finite-dimensional representations of the fundamental group $\pi_1(X)$. To see this, first define locally constant sheaves and state their equivalence with covers and $\pi_1(X)$-sets as in definition 2.5.6, theorem 2.5.9, and theorem 2.3.4 of [20]. Don't give any proofs; just construct the fiber functor. Note that Szamuely uses Deligne's convention that for loops $\alpha, \beta \in \pi_1(X)$ the composition $\alpha\beta$ means first β, then α. You should make the world a better place and do the same! Then explain corollary 2.6.2.

Now let $X/\mathbb{C}$ be a smooth variety. Local systems on X^{an} are closely related to covers of X^{an}. Unfortunately, the category of covers of X^{an} does not seem to admit an algebraic description. Nevertheless we can describe the full subcategory of finite covers. Remark that if $s\colon Z \to X^{an}$ is a cover, one can endow Z with a complex structure making s holomorphic. Moreover, in that case s is a local isomorphism by the inverse function theorem, i.e. s is an isomorphism everywhere locally on Z. Grothendieck made the essential observation that the algebraic analogue of local isomorphisms are étale morphisms. To be precise, a morphism of complex varieties $Y \to X$ is étale if and only if $Y^{an} \to X^{an}$ is a local isomorphism. Of course, not every local isomorphism is a topological cover; but if $Y \to X$ is finite étale, then $Y^{an} \to X^{an}$ is a finite cover. This will give an equivalence between finite étale maps $Y \to X$ and finite covers $Z \to X^{an}$.

Let's make things precise. Explain without proof theorem 5.10 in [15]. Then define the fiber functor and explain that the category of finite étale covers of X is equivalent to that of finite π-sets for some profinite group π. We call $\pi = \pi_1^{et}(X)$ the (étale) fundamental group of X. Follow theorem 5.24 in [15] or theorem 5.4.2 in [20]. In either case, there is time only for a sketch of the proof. Also state [20], theorem 5.7.4.

At last return to local systems. A local system on X^{an} is a sheaf of vector spaces $\mathcal{V}$ on X^{an} that admits a cover $s\colon Z \to X^{an}$ such that $s^{-1}\mathcal{V}$ is constant. Say the system is finite if we can take s to be finite. Similarly, define a finite local system on X to be a sheaf of vector spaces $\mathcal{V}$ on X that admits a finite étale cover $s\colon Y \to X$ such that $s^{-1}\mathcal{V}$ is constant. By the above reasoning the categories of analytic and algebraic finite local systems are equivalent. Moreover, analogous to corollary 2.6.2 of [20] algebraic finite local systems are equivalent to finite-dimensional represenions of $\pi_1^{et}(X)$. The induced functor from $\pi_1^{et}(X)$-representations to $\pi_1(X^{an})$-representations is simply restriction along $\pi_1(X^{an}) \to \hat{\pi}_1(X^{an}) = \pi_1^{et}(X)$.

06-06. The derived category of $\mathcal{D}$-modules **Mara**

Recall quickly, without proofs, the definition of sheaf cohomology via right derived functors. More precisely, define injective objects and injective resolutions; state that on any scheme X the category of $\mathcal{O}_X$-modules has enough injectives; and if $\mathcal{F}$ is an $\mathcal{O}_X$-module with injective resolution $\mathcal{I}^\bullet$, then $\mathrm{H}^i(X, \mathcal{F}) = \Gamma(X, \mathcal{I}^i) = \mathcal{I}^i(X)$. See sections III.1–2 of [11] for more details.

Show that the category of $\mathcal{O}_X$-quasi-coherent $\mathcal{D}_{X/k}$-modules on a smooth variety X/k has enough injectives following proposition VI.2.1 in [2]. Borel denotes this category by $\mu(\mathcal{D}_X)$. You may use without proof that, for any not necessarily commutative ring R, the category of left R-modules has enough injectives. Similarly show that there are enough projectives following proposition VI.2.4. For both parts a sketch of the proofs suffices.

During the remainder of the talk, construct $\mathrm{D}_{\mathrm{qc}}(\mathcal{D}_{X/k})$, the derived category of the category of $\mathcal{O}_X$-quasi-coherent $\mathcal{D}_{X/k}$-modules. In fact, construct the derived category $\mathrm{D}(\mathcal{A})$ of any abelian category $\mathcal{A}$. The emphasis should not be on proofs, but rather on how we can work in practice with objects and morphisms of $\mathrm{D}(\mathcal{A})$. Also do not mention the triangulated structure on $\mathrm{D}(\mathcal{A})$. A good reference is [21], items 10.1.1–2, 10.3.1–11 without the set-theoretic remarks, 10.3.15–16, and the special case $\mathrm{K} = \mathrm{K}(\mathcal{A})$, $S = \{\text{quasi-isomorphisms}\}$ of proposition 10.4.1(1).

13-06. Functoriality of $\mathcal{D}$-modules — Elena

Introduce derived functors between derived categories as in [21], 10.5.1–7. Replace corollary 10.5.7 by the following equivalent statement. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories and assume $\mathcal{A}$ has enough injectives. Each bounded below complex $A^\bullet$ in $\mathcal{A}$ admits an injective resolution $A^\bullet \to I^\bullet$ and $\mathrm{H}^i(\mathrm{R}F(A^\bullet)) = \mathrm{H}^i(F(I^\bullet))$. Similarly for $\mathrm{L}F$.

In this talk we study two types of functoriality for $\mathcal{D}$-modules: pullback and pushforward. First pullback. Let $f\colon X \to Y$ be a morphism of smooth k-varieties and $\mathcal{F}$ a $\mathcal{D}$-module on Y. The usual inverse image $f^*\mathcal{F}$ has a natural $\mathcal{D}_{X/k}$-module structure. We denote this $\mathcal{D}$-module by $f^+\mathcal{F}$. Note that f^+, like f^*, is right exact but in general not exact. So we get a derived functor $\mathrm{L}f^+\colon \mathrm{D}^-_{\mathrm{qc}}(\mathcal{D}_{Y/k}) \to \mathrm{D}^-_{\mathrm{qc}}(\mathcal{D}_{X/k})$. One has $\mathrm{L}f^+\mathcal{D}_{Y/k} = f^+\mathcal{D}_{Y/k}$ and for any $\mathcal{F}^\bullet \in \mathrm{D}^-_{\mathrm{qc}}(\mathcal{D}_{Y/k})$ the complex of $\mathcal{O}_X$-modules underlying $\mathrm{L}f^+\mathcal{F}^\bullet$ is just $\mathrm{L}f^*\mathcal{F}^\bullet$. See §1.3 and §1.5 of [12] for details; be warned that they write f^* for our f^+.

Next pushforward. Up to now, all $\mathcal{D}$-module were left $\mathcal{D}$-modules. Pushforward will be defined for right $\mathcal{D}$-modules. In characteristic zero that is not a problem, since then there is an equivalence between left and right $\mathcal{D}$-modules. Indeed, if $\operatorname{char} k = 0$, we know that $\mathcal{D}_{X/k}$ is the universal enveloping algebra of $\mathcal{T}_{X/k}$. Hence the canonical bundle $\omega_{X/k}$ is a right $\mathcal{D}$-module as in [12], page 19. (This text writes $\Omega_{X/k}$ for $\omega_{X/k}$.) Explain proposition 1.2.12 from [12]. Take care to define the correct $\mathcal{D}$-module structures from proposition 1.2.9.

If $\mathcal{E}$ is a right $\mathcal{D}$-module on X, its direct image is $f_+\mathcal{E} = f_*(\mathcal{E} \otimes_{\mathcal{D}_{X/k}} f^+\mathcal{D}_{Y/k})$. Deriving this functor is somewhat complicated since f_* is left exact whereas $\otimes$ is right exact. That is resolved using the Spencer complex $\mathrm{Sp}^\bullet_{X\to Y}$, a flat resolution of $f^+\mathcal{D}_{Y/k}$ in $\mathcal{D}_{X/k}$-modules. Then $\mathrm{R}f_+\colon \mathrm{D}^+_{\mathrm{qc}}(\mathcal{D}_{X/k}) \to \mathrm{D}^+_{\mathrm{qc}}(\mathcal{D}_{Y/k})$ is given by $\mathcal{E}^\bullet \mapsto \mathrm{R}f_*(\mathcal{E}^\bullet \otimes_{\mathcal{D}_{X/k}} \mathrm{Sp}^\bullet_{X\to Y})$. The point is that we don't need to derive the tensor product anymore. For details see items 1.4.2–3 and 3.3.1–3 of [18]. Finally, if $\operatorname{char} k = 0$ describe the analogue for left modules following page 23 in [12].

20-06. No seminar

27-06. The Gauss–Manin connection — Efstathia

Throughout, let k have characteristic zero and $f\colon X \to Y$ a morphism of smooth k-varieties. Recall the construction of the derived direct image $\mathrm{R}f_+\mathcal{E}$ of a left $\mathcal{D}$-module $\mathcal{E}$ on X. Here is an alternative construction. Associated to a flat connection $\mathcal{E}$ on X we have the de Rham complex $\Omega^\bullet_{X/k}(\mathcal{E})$. See definition 1.4.1 in [18] and prove propositions 1.4.3 and 1.4.4. Then explain the alternative construction of $\mathrm{R}f_+$ in proposition 3.3.5.

In the special case $\mathcal{E} = \mathcal{O}_X$ we call the connections $\mathcal{H}^i(\mathrm{R}f_+\mathcal{O}_X)$ the Gauss–Manin connections with respect to f. They can be computed using a spectral sequence: see §3.1 of [1]. You can say a few words about that if you want. During the remainder of your talk, you will compute the Gauss–Manin connection if Y is a curve. Then the spectral sequence reduces to the boundary map in a certain exact sequence. Explain this following [1], §3.3. Omit the Čech cohomology computations on page 11 but give as many details as possible for example 3.4.

04-07. *F*-divided bundles **Wouter**

Recall the Riemann–Hilbert correspondence: on a proper smooth variety $V/\mathbb{C}$ there is an equivalence of categories between stratified bundles on V and finite-dimensional complex representations of the topological fundamental group $\pi_1(V^{\mathrm{an}})$. Deduce theorem 0.4 of [9]. The proof uses the main result from [10]. Depending on time, you may use that as a black box.

Gieseker's conjecture states that roughly the same result should hold for varieties over a field of positive characteristic. In the next few talks we study this conjecture. So from now on suppose that $\operatorname{char} k = p$ is positive. Then stratified bundles have an alternative description as F-divided bundles. Define F-divided bundles and their morphisms as in [9], definitions 1.1–2. Prove theorem 1.3 and end your talk with the proof of proposition 1.7.

Here follow some hints regarding the terminology in [9]. Ignoring Gieseker, take for V/k just a smooth variety, not a formal scheme. The Frobenius endomorphism $F\colon V \to V$ is the absolute Frobenius, i.e. the identity on topological spaces but with $F^{\#}\colon \mathcal{O}_V \to \mathcal{O}_V$ the p-power map. If $\varphi\colon k \to k$ is the usual p-Frobenius, there is a commutative (not cartesian) diagram

$$\begin{array}{ccc} V & \xrightarrow{F} & V \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\varphi^*} & \operatorname{Spec} k. \end{array}$$

A map of bundles $\sigma\colon \mathcal{E} \to \mathcal{F}$ is p-linear if for all $f \in \mathcal{O}_V$ and $x \in \mathcal{E}$ one has $\sigma(fx) = f^p\sigma(x)$. Gieseker's flat bundles are nowadays known as F-divided bundles. Please use the modern term. What Gieseker calls horizontal maps of stratified bundles we just call morphisms.

11-07. The Gieseker conjecture **Yun**

First prove theorem 1.8 of [9] as a corollary to proposition 1.7 which was proved in the previous talk. (If otherwise your talk will be too long, you may skip this part.)

Explain the construction above proposition 1.9 in [9]. Then prove proposition 1.9 and theorem 1.10. State Gieseker's conjecture: the converse of theorem 1.10 should hold for projective smooth varieties. The conjecture has recently been proven in [6, 7]. If you feel confident, you can give a short sketch of the proof following the introduction of [6].

To finish your talk, verify that the Gieseker conjecture holds for $\mathbb{P}^n$. That is, show that $\mathbb{P}^n$ is simply connected and prove theorem 2.2 of [9]. Here is a proof of $\pi_1(\mathbb{P}^n) = 1$. For $n = 1$ see example IV.2.5.3 in [11]. For $n \geq 2$ suppose $U \to \mathbb{P}^n$ is a connected finite étale cover. Let $H \subset \mathbb{P}^n$ be a hyperplane. Then $U' = U \times_{\mathbb{P}^n} H$ is connected as well by the Lefschetz hyperplane theorem; see e.g. [11], corollary III.7.9. Applying induction to $H \cong \mathbb{P}^{n-1}$ the cover $U \to \mathbb{P}^n$ has degree 1 above H, but then it must have degree 1 everywhere.

18-07. $\mathcal{D}$-modules on K3 surfaces and unirational varieties **Tanya**

In this talk, we verify the Gieseker conjecture for two classes of examples: K3 surfaces and unirational varieties. First K3 surfaces. Recall the definition of a K3 surface and show that K3 surfaces are simply connected following [13], remark 2.4. (Don't spend too much time on this.) Then prove, in positive characteristic, theorem 2.3 of [9].

Unirational varieties require some preparation. First prove items 2.7–10 of [9]. For proposition 2.7 you can follow the easier proof in [14], proposition 2.7(c). Now recall that a variety X/k is unirational if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$. Then its function field has a finite extension isomorphic to $k(x_1, \ldots, x_n)$. So Gieseker's conjecture for unirational varieties follows from theorems 2.2 and 2.10. Note that there exist examples due to Shioda of unirational surfaces with non-trivial fundamental group.

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