The étale fundamental group

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1. Topology

Let X be a connected topological space. Let $x \in X$ be a point. An important invariant of (X, x) is the *(topological) fundamental group*

$$\pi(X, x) := \{\text{loops } x \rightsquigarrow x \text{ in } X\}/\simeq.$$

It can also be described in terms of covers. A *cover* of X is a map $p\colon Y \to X$ such that every point $x \in X$ has an open neighborhood $U \subseteq X$ with $p^{-1}(U) \cong U \times p^{-1}(x)$ as spaces over U (endowing $p^{-1}(x)$ with the discrete topology). A cover $Y \to X$ is *universal* if Y is simply connected. In this case $\pi(X, x) = \operatorname{Aut}_X Y$.

Theorem 1.1. *Suppose X admits a universal cover. Then the functor*

$$\operatorname{Cov} X \to \pi(X, x)\text{-Set}, \qquad p \mapsto p^{-1}(x)$$

is an equivalence. ♦

Theorem 1.2. *There is a profinite group π, unique up to isomorphism, such that*

$$\operatorname{FCov} X \approx \pi\text{-FSet}.$$

If X admits a universal cover, then π is isomorphic to the profinite completion $\hat{\pi}(X, x)$. ♦

All data in this theorem can be made functorial in (X, x).

Example 1.3. The circle S^1 has fundamental group $\pi(S^1, x) = \mathbb{Z}$. It has the universal cover $\mathbb{R} \to S^1$, $t \mapsto \exp 2\pi i t$, with automorphism group generated by the shift $t \mapsto t + 1$. In the setting of theorem 1.2, suppose A is a transitive finite $\hat{\mathbb{Z}}$-set. Then $A \cong \mathbb{Z}/n\mathbb{Z}$, and it corresponds to the finite cover $\mathbb{R}/n\mathbb{Z} \to S^1$, $t \mapsto \exp 2\pi i t$. ♦

2. Algebraic geometry

Let X be a connected scheme. Let $x \in X$ be a point. The topological fundamental group $\pi(X, x)$ is not a useful invariant, due to the Zariski topology. As usual, the correct notion of a covering in algebraic geometry is an étale map. Then theorem 1.2 has the following analogue.

Theorem 2.1. *There is a profinite group π, unique up to isomorphism, such that*

$$\operatorname{FEt} X \approx \pi\text{-FSet}.$$ ♦

Given a geometric point $\bar{x}$ of X, we can define π and the equivalence functorially in $(X, \bar{x})$. It is the *étale fundamental group* $\pi^{\text{et}}(X, \bar{x})$.

Often $\pi^{\text{et}}(X, \bar{x})$ is the desired analogue of the topological fundamental group. This can be seen for instance in the complex case: if X is a connected complex variety and x a closed point, then $\pi^{\text{et}}(X, \bar{x}) = \hat{\pi}(X^{\text{an}}, x)$.

Example 2.2. Let X be the complex projective line with 0 and ∞ identified. Its analytification is the Riemann sphere S^2 with two points identified, hence $\pi(X^{\mathrm{an}}, x) = \mathbb{Z}$. We get $\pi^{\mathrm{et}}(X, \bar{x}) = \hat{\mathbb{Z}}$. In the setting of theorem 2.1, suppose $A \cong \mathbb{Z}/n\mathbb{Z}$ is a transitive finite $\hat{\mathbb{Z}}$-set. It corresponds to the finite étale X-scheme consisting of n copies of $\mathbb{P}^1$, where 0 in the i^{th} copy is identified with ∞ in the $(i+1)^{\text{st}}$ copy, cyclically. ♦

Remark 2.3. In the preceding example, there is a natural 'universal' étale X-scheme, with automorphism group $\mathbb{Z}$. It would be nice if one could actually detect this. This 'defect' is repaired by the *pro-étale fundamental group*, to be introduced next week. ♦

3. Galois theory

The formalism behind theorems 1.2 and 2.1 is a type of Galois theory. It is used to classify categories of the form π-FSet for some profinite group π.

Definition 3.1. Let $\mathcal{C}$ be a category and $F\colon \mathcal{C} \to \mathrm{FSet}$ a functor. Then $\mathcal{C}$ is a *Galois category* with *fundamental functor F* if

- $\mathcal{C}$ has finite limits and colimits,
- any map $f\colon X \to Y$ in $\mathcal{C}$ can be written as $f = m \circ e$ with e an epimorphism and m a monomorphism onto a direct summand of Y, and
- F is exact and conservative. ♦

Example 3.2. Let π be a profinite group. Then π-FSet with the forgetful functor $\pi\text{-FSet} \to \mathrm{FSet}$ is a Galois category. (We will see that, up to equivalence, this is the only example.) ♦

Let $(\mathcal{C}, F)$ be a Galois category. Consider the automorphism group $\operatorname{Aut} F$. Endowing each finite permutation group $S(F(X))$ with the discrete topology, the subgroup $\operatorname{Aut} F \subseteq \prod_{X \in \mathcal{C}} S(F(X))$ is closed. In fact $\operatorname{Aut} F$ is profinite. The action of $\operatorname{Aut} F$ on each $F(X)$ is continuous. So we get a functor $\mathcal{C} \to \operatorname{Aut} F\text{-FSet}$.

Theorem 3.3. *Let $(\mathcal{C}, F)$ be a Galois category.*

- *The functor $\mathcal{C} \to \operatorname{Aut} F\text{-FSet}$ is an equivalence.*
- *Let π be a profinite group. If F factors over an equivalence $\mathcal{C} \to \pi\text{-FSet}$, then $\pi = \operatorname{Aut} F$.* ♦

Moreover, the group $\operatorname{Aut} F$ does not really depend on F.

Theorem 3.4. *Let $\mathcal{C}$ be a category.*

- *If $F, F'\colon \mathcal{C} \to \mathrm{FSet}$ both make $\mathcal{C}$ into a Galois category, then $F \cong F'$.*
- *Let π, π' be profinite groups. If $\mathcal{C}$ is equivalent to both π-FSet and π'-FSet, then $\pi \cong \pi'$.* ♦

4. Applications

From the preceding theory we can easily prove theorems 1.2 and 2.1. For the first, let (X, x) be a pointed connected topological space. We define the fiber functor $F_x\colon \mathrm{FCov}\, X \to \mathrm{FSet}$, $p \mapsto p^{-1}(x)$.

Lemma 4.1. *Let X be a topological space. Let $p\colon Y \to X$ and $q\colon Z \to X$ be finite coverings, and $f\colon Y \to Z$ a morphism of coverings. Then each $x \in X$ has an open neighborhood $U \subseteq X$ where p and q are trivial, such that f is of the form $\mathrm{id}_U \times \alpha\colon U \times p^{-1}(U) \to U \times q^{-1}(U)$ above U.* ♦

Theorem 4.2. *The pair $(\mathrm{FCov}\, X, F_x)$ is a Galois category.* ♦

In the algebraic geometry setting, we do essentially the same. Let $(X, \bar{x})$ be a geometrically pointed connected scheme. Let $F_{\bar{x}}$ be the fiber functor $\mathrm{FEt}\, X \to \mathrm{FEt}\, \bar{x} \to \mathrm{FSet}$.

Theorem 4.3. *The pair* $(\mathrm{FEt}\, X, F_{\bar{x}})$ *is a Galois category.* ♦

It is a good exercise to prove this theorem in the case $X = \mathrm{Spec}\, k$, where k is a field. Observe that then $\pi^{\mathrm{et}}(X, \bar{x}) = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. This illustrates the terminology 'Galois theory'.

References

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[2] H.W. Lenstra, *Galois theory for schemes*. `http://websites.math.leidenuniv.nl/algebra/GSchemes.pdf`, 2008.