Fundamental groups

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These are (incomplete) notes for a crash course held at FU Berlin.

1. Motivation

The topological fundamental group

Let X be a connected topological space and $x \in X$ a point. A well-known invariant of X is the *fundamental group*

$$\pi(X, x) := \{\text{paths } x \rightsquigarrow x \text{ in } X\}/\simeq.$$

It has an alternative description in terms of coverings. A *covering* of X is a map $p\colon Y \to X$ such that each point $x \in X$ has an open neighborhood U with $p^{-1}(U) = U \times p^{-1}(x)$ over U, considering $p^{-1}(x)$ as a discrete space. We write $\operatorname{Cov} X$ for the category of coverings of X. A covering $Y \to X$ is *universal* if Y is simply connected.

Exercise 1A. Show that any two universal coverings of X are isomorphic. Are they necessarily uniquely isomorphic? ⧫

Exercise 1B. Let $p\colon Y \to X$ be a covering. Show that $\pi(X, x)$ has a natural action on the fiber $p^{-1}(x)$. If p is universal, show that the action extends to one on Y over X and that the map $\pi(X, x) \to \operatorname{Aut}_X Y$ is an isomorphism. ⧫

The following is a characterization of $\pi(X, x)$, assuming X admits a universal covering. That is true at least when X is locally simply connected, i.e. has a basis of simply connected opens.

Theorem 1.1. *Suppose X admits a universal covering. The functor*

$$F_x\colon \operatorname{Cov} X \to \pi(X, x)\text{-Set}, \quad p \mapsto p^{-1}(x)$$

is an equivalence. ⧫

We omit the proof.

Exercise 1C. Show that theorem 1.1 characterizes $\pi(X, x)$ in the following sense: if G and H are groups such that the categories G-Set and H-Set are equivalent, then $G \cong H$. ⧫

The *degree* of a covering $p\colon Y \to X$ at a point $x \in X$ is the cardinality of the fiber $p^{-1}(x)$. This is locally constant on X, hence by connectedness constant. We call a covering *finite* if its degree (at any point) is finite. Consider the category $\operatorname{FCov} X$ of finite coverings. Under F_x it corresponds to the category $\pi(X, x)$-FSet of finite $\pi(X, x)$-sets.

Theorem 1.2. *There is a canonical profinite group $\hat{\pi}(X, x)$ and an equivalence of categories*

$$F_x\colon \operatorname{FCov} X \to \hat{\pi}(X, x)\text{-FSet}, \quad p \mapsto p^{-1}(x).$$

If X admits a universal covering, $\hat{\pi}(X, x)$ is the profinite completion of $\pi(X, x)$. ⧫

Note here that actions of topological groups are always assumed continuous.

Although the statement is weaker than in theorem 1.1, no assumption on X is required. The proof will be given later using Grothendieck's Galois formalism.

Exercise 1D. Let G be a group and $\hat{G}$ its profinite completion. Show that the categories G-FSet and $\hat{G}$-FSet are isomorphic. ⧫

The Galois group

Let K be a field and $\bar{K}$ a separable closure. A well-known invariant of K is the *Galois group* $\operatorname{Gal}(\bar{K}/K)$. It may be defined as a profinite group by

$$\operatorname{Gal}(\bar{K}/K) := \lim_{L} \operatorname{Aut}_K L$$

where the limit runs over all intermediate fields $K \subseteq L \subseteq \bar{K}$ that are finite Galois over K. The fundamental theorem of Galois theory says that in fact $\operatorname{Gal}(\bar{K}/K) = \operatorname{Aut}_K \bar{K}$ and that the intermediate fields of $\bar{K}/K$ that are finite over K correspond to the open subgroups of $\operatorname{Gal}(\bar{K}/K)$.

Let us enhance this a bit. A K-algebra A is called *finite separable* if $A \cong L_1 \times \ldots \times L_n$ for some finite separable field extensions $L_1, \ldots, L_n$ of K. Write $\operatorname{FSep} K$ for the category of finite separable K-algebras.

Exercise 1E. Let A be a finite separable K-algebra. Show that $\operatorname{Hom}_K(A, \bar{K})$ is a finite set with a natural action of $\operatorname{Gal}(\bar{K}/K)$. ♦

Theorem 1.3. *The contravariant functor*

$$F_{\bar{K}} \colon \operatorname{FSep} K \to \operatorname{Gal}(\bar{K}/K)\text{-FSet}, \quad A \mapsto \operatorname{Hom}_K(A, \bar{K})$$

is an anti-equivalence. ♦

This is easily proven from the fundamental theorem of Galois theory. However, we will see it later as a consequence of Grothendieck's Galois formalism.

Exercise 1F. Characterize the finite $\operatorname{Gal}(\bar{K}/K)$-sets that correspond under $F_{\bar{K}}$ to fields. ♦

Exercise 1G. Let L/K be a Galois extension. A finite separable K-algebra A is *split* over L if $L \otimes_K A \cong L^n$ for some $n \in \mathbb{N}$. Show that $F_{\bar{K}}$ induces an anti-equivalence between the categories of finite separable K-algebras split over L and $\operatorname{Gal}(L/K)$-FSet. ♦

2. Galois categories

In topology, the fundamental group describes (finite) coverings of a topological space. In field theory, the Galois group describes finite extensions of a field. There are many similarities between the constructions: e.g. both arise as an automorphism group, and the choice of base point has the same function as the choice of separable closure. These examples represent two extremes of Grothendieck's Galois theory.

Some category theory

Here is the main definition.

Definition 2.1. A *Galois category* is pair $(\mathcal{C}, F)$ where $\mathcal{C}$ is a category and $F \colon \mathcal{C} \to \operatorname{FSet}$ a functor, called the *fundamental* or *fiber functor*, such that

- $\mathcal{C}$ has finite limits and finite colimits,
- F is exact and conservative,
- each map f in $\mathcal{C}$ has a factorization $f = hg$ with g an epi- and h a monomorphism, and
- each subobject in $\mathcal{C}$ admits a complement. ♦

Example 2.2. Let π be a profinite group. We will see that the category π-FSet of finite continuous π-sets, with the forgetful functor $F_\pi \colon \pi\text{-FSet} \to \text{FSet}$, is a Galois category. ♦

We explain the terminology. Let $X \colon \mathcal{I} \to \mathcal{C}$ a functor. A *limit* over X, denoted $\lim X = \lim_{i \in \mathcal{I}} X_i$, is an object $Y \in \mathcal{C}$ together with maps $f_i \colon Y \to X_i$ satisfying $f_i = X(\varphi) f_j$ for each map $\varphi \colon j \to i$ of $\mathcal{I}$, and universally so: for any other such Z and maps $g_i \colon Z \to X_i$ there exists a unique map $h \colon Z \to Y$ with $g_i = f_i h$ for all $i \in \mathcal{I}$.

$$\begin{tikzcd}
 & & X_j \arrow[dd, "X(\varphi)"] \\
Z \arrow[urr, bend left, "g_j"] \arrow[drr, bend right, "g_i"'] \arrow[r, dashed, "\exists!\, h"] & Y \arrow[ur, "f_j"] \arrow[dr, "f_i"'] & \\
 & & X_i
\end{tikzcd}$$

So giving a map $Z \to \lim X$ is the same as giving maps $Z \to X_i$ compatible as above with the transformations $X(\varphi)$. In formulas,

$$\operatorname{Hom}(Z, \lim X) = \lim_{i \in \mathcal{I}} \operatorname{Hom}(Z, X_i).$$

Conversely, a *colimit* over X is a universal object $\operatorname{colim} X$ together with maps $f_i \colon X_i \to \operatorname{colim} X$ satisfying $f_i = f_j X(\varphi)$ for each map $\varphi \colon i \to j$ of $\mathcal{I}$. It satisfies

$$\operatorname{Hom}(\operatorname{colim} X, Z) = \lim_{i \in \mathcal{I}^{\mathrm{op}}} \operatorname{Hom}(X_i, Z).$$

Limits and colimits are uniquely unique, if they exist.

We say $\mathcal{I}$ is *cofiltered* if for all $i, i' \in \mathcal{I}$ there exist $j \in \mathcal{I}$ and maps $j \to i, j \to i'$, and for all maps $\varphi, \varphi' \colon j \to i$ in $\mathcal{I}$ there exists a map $\psi \colon k \to j$ with $\varphi\psi = \varphi'\psi$. Cofiltered limits are particularly well-behaved. For instance, if $\mathcal{I}$ is cofiltered and $X \colon \mathcal{I} \to \text{Set}$ a functor such that all transition maps $X(\varphi)$ are surjective, the projections $\lim X \to X_i$ are surjective as well. Dually, if $\mathcal{I}$ is *filtered* and all transition maps are injective, then so are the coprojections $X_i \to \operatorname{colim} X$.

A subcategory $I \colon \mathcal{J} \hookrightarrow \mathcal{I}$ is *initial* if for all $i \in \mathcal{I}$ there exists a map $j \to i$ with $j \in \mathcal{J}$ and for all maps $\varphi \colon j \to i$ and $\varphi' \colon j' \to i$ in $\mathcal{I}$ with $j, j' \in \mathcal{J}$ there exist maps $\psi \colon k \to j$ and $\psi' \colon k \to j'$ in $\mathcal{J}$ with $\psi\varphi = \psi'\varphi'$. In this case it is easily verified that $\lim X = \lim XI$. Dually, if $I \colon \mathcal{J} \hookrightarrow \mathcal{I}$ is *final* one has $\operatorname{colim} X = \operatorname{colim} XI$.

A (co)limit $X \colon \mathcal{I} \to \mathcal{C}$ is *finite* if $\mathcal{I}$ has finitely many objects and morphisms. Certain types of finite limits are of special interest: the *final object* (case $\mathcal{I} = \varnothing$), *equalizers* (case $\mathcal{I} = \{\bullet \rightrightarrows \bullet\}$), and *fiber products* (case $\mathcal{I} = \{\bullet \to \bullet \leftarrow \bullet\}$). The dual colimits are the *initial object*, *coequalizers* and *pushouts*. One can show that a category has all finite limits if and only if it has a final object and all fiber products. Dually a category has all finite colimits if and only if it has an initial object and all pushouts.

Exercise 2A. Show that π-FSet has all finite limits and finite colimits. ♦

A functor $F \colon \mathcal{C} \to \mathcal{D}$ is *left exact* if it commutes with finite limits, i.e. for any finite $\mathcal{I}$ and $X \colon \mathcal{I} \to \mathcal{C}$ one has $F(\lim X) = \lim FX$. It is *right exact* if it commutes with finite colimits and *exact* if it commutes with both. This is the case precisely if F preserves the final and inital object, commutes with fiber products, and commutes with pushouts.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *conservative* or *reflects isomorphisms* if for each map f of $\mathcal{C}$, if $F(f)$ is an isomorphism then so is f. (The converse always holds.) In case of an exact conservative functor, something more is true: F also reflects (co)limits, i.e. $Y = \lim X$ if and only if $F(Y) = \lim FX$ and analogously for colimits.

Exercise 2B. Show that the forgetful functor $F_\pi\colon \pi\text{-FSet} \to \text{FSet}$ is exact and conservative. ♦

A map $f\colon Y \to X$ is a *monomorphism* if $fg = fh$ implies $g = h$ for all $g, h\colon Z \to Y$. Equivalently, the diagonal $Y \to Y \times_X Y$ is an isomorphism. So if $F\colon \mathcal{C} \to \mathcal{D}$ is exact and conservative, f is a monomorphism if and only if $F(f)$ is. The dual notion is an *epimorphism*.

A *subobject* of $X \in \mathcal{C}$ is a monomorphism $Y \to X$. If two subobjects are isomorphic, they are uniquely isomorphic and we consider them the same subobject. A *complement* of $Y \to X$ is a second subobject $Z \to X$ such that the natural map $Y \sqcup Z \to X$ is an isomorphism.

Exercise 2C. Finish the proof that $(\pi\text{-FSet}, F_\pi)$ is a Galois category. ♦

Functors also live in categories: the functor category $\text{Func}(\mathcal{C}, \mathcal{D})$ whose arrows are the natural transformations. So any functor $F\colon \mathcal{C} \to \mathcal{D}$ has an automorphism group $\operatorname{Aut} F$. Note that an automorphism of F consists of permutations α_X of $F(X)$ for all $X \in \mathcal{C}$, such that for all $f\colon X \to Y$ one has a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & F(X) \arrow[d, "F(f)"] \\ F(Y) \arrow[r, "\alpha_Y"] & F(Y). \end{tikzcd}$$

Lemma 2.3. *Let $F\colon \mathcal{C} \to \text{FSet}$ be a functor. Then $\operatorname{Aut} F$ is canonically a profinite group.*

Proof. Let $\mathrm{S}(P)$ be the permutation group of a set P. As explained above, we have

$$\operatorname{Aut} F \subseteq \prod_{X \in \mathcal{C}} \mathrm{S}(F(X)).$$

Endow each group $\mathrm{S}(F(X))$ with the discrete topology. The product $\prod_{X \in \mathcal{C}} \mathrm{S}(F(X))$ is profinite. We claim that $\operatorname{Aut} F$ is a closed subgroup, hence profinite as well. But indeed $\operatorname{Aut} F$ is the intersection of the subsets $\{(\alpha_X)_{X \in \mathcal{C}}\colon F(f)\alpha_Y = \alpha_Z F(f)\}$ over all $f\colon Y \to Z$, and those subsets are closed. ∎

Exercise 2D. Show that the sets $\{\alpha \in \operatorname{Aut} F\colon \alpha_X = 1\}$ ranging over $X \in \mathcal{C}$ form a basis of open neighborhoods of $1 \in \operatorname{Aut} F$. ♦

Exercise 2E. Let $F_\pi\colon \pi\text{-FSet} \to \text{FSet}$ be the forgetful functor. Show that $\operatorname{Aut} F_\pi = \pi$. ♦

Exercise 2F. Let G be any discrete or topological group. Show that $(G\text{-FSet}, F_G)$ is a Galois category. Show that $\operatorname{Aut} F_G$ is the profinite completion of G. ♦

The Galois correspondence

For each $X \in \mathcal{C}$ we have a projection map $\operatorname{Aut} F \to \mathrm{S}(F(X))$. This action of $\operatorname{Aut} F$ on $F(X)$ is continuous and functorial, so F extends to a functor $\mathcal{F}\colon \mathcal{C} \to \operatorname{Aut} F\text{-FSet}$. The main result of Grothendieck's Galois theory is that $\mathcal{F}$ is a 'Galois correspondence' between $\mathcal{C}$ and $\operatorname{Aut} F\text{-FSet}$. One may call $\operatorname{Aut} F$ the *Galois* or *fundamental group* of $\mathcal{C}$.

Theorem 2.4. *Let $(\mathcal{C}, F)$ be a Galois category. The functor $\mathcal{F} \colon \mathcal{C} \to \operatorname{Aut} F\text{-FSet}$ is an equivalence.*

Proof. We proceed in several steps. To begin with, an object $X \in \mathcal{C}$ is *connected* if it has precisely two subobjects. If $Y \to X$ is a subobject other than the initial object and X, it has a complement $Z \to X$ and $F(X) = F(Y) \sqcup F(Z)$ with $F(Y), F(Z) \neq \varnothing$. By induction to $\#F(X)$ we see that X is a coproduct of connected subobjects. This decomposition in *connected components* is unique up to ordering, and the subobjects of X are coproducts of connected components.

Let C be connected and $c \in F(C)$. For any $X \in \mathcal{C}$ the map

$$(*) \qquad \operatorname{Hom}(C, X) \to F(X), \quad f \mapsto F(f)(c)$$

is injective. Indeed, if $F(f)(c) = F(g)(c)$, then the equalizer B of f and g (the largest subobject $B \to C$ on which f and g coincide) has $c \in F(B)$. By connectedness, $B = C$ and $f = g$.

In particular $\operatorname{Aut} C \subseteq \operatorname{Hom}(C, C)$ is finite. If $G \subseteq \operatorname{Aut} C$ is a subgroup, the quotient C/G is the coequalizer of all $\sigma \in G$, hence exists. We say C is *Galois* if C is connected and $C/\operatorname{Aut} C$ is the final object. Equivalently, $\operatorname{Aut} C$ acts transitively on $F(C)$. As $\#\operatorname{Aut} C \leq \#F(C)$ by connectedness, for Galois objects we have $\#\operatorname{Aut} C = \#F(C)$ and the $\operatorname{Aut} C$-action on $F(C)$ is free as well.

Take $X \in \mathcal{C}$. We set $D := \prod_{x \in F(X)} X$ and $c := \mathrm{id}_{F(X)} \in F(D) = \prod_{x \in F(X)} F(X)$. Let $C \to D$ be the connected component with $c \in F(C)$. For $x \in F(X)$ let $f_x \colon C \to X$ be the projection on the x'th factor. Then $F(f_x)(c) = x$, hence $(*)$ is bijective for these C, c and X. We claim moreover that C is Galois. Let $c' \in F(C)$. Then $\operatorname{Hom}(C, X) \to F(X), f \mapsto F(f)(c')$ is an injection of equipotent finite sets, hence surjective. As $\operatorname{Hom}(C, X) = \{f_x \colon x \in F(X)\}$, this means that $c' \in \prod_{x \in F(X)} F(X)$ is a permutation. Let σ be the induced automorphism of D by permuting the factors. It sends c to c' and C to a connected component $C' \to D$. We have $c' \in F(C) \cap F(C')$ so $C = C'$ and σ restricts to an automorphism of C. We proved that for any X there exists a Galois object C and $c \in F(C)$ such that $(*)$ is a bijection.

If X is connected, something more is true. Firstly, the right action of $\operatorname{Aut} C$ on $\operatorname{Hom}(C, X)$ is transitive. Indeed, take $f, g \colon C \to X$. From the existence of epi-mono-factorizations and connectedness of X we find that $F(f)$ is surjective. Choose $d \in F(C)$ with $F(f)(d) = F(g)(c)$. Choose $\sigma \in \operatorname{Aut} C$ with $F(\sigma)(c) = d$. Then $F(f\sigma)(c) = F(g)(c)$ hence $f\sigma = g$.

Now fix a map $f \colon C \to X$ and define $G := \{\sigma \in \operatorname{Aut} C \colon f\sigma = f\}$. We claim that the induced map $\bar{f} \colon C/G \to X$ is an isomorphism. It suffices to show that $F(\bar{f}) \colon F(C)/G \to F(X)$ is a bijection. The latter is certainly surjective because $F(f)$ is. Since G acts freely on $F(C)$ we have $\#(F(C)/G) = \#F(C)/\#G = [\operatorname{Aut} C : G]$. On the other hand $\operatorname{Hom}(C, X)$ has a transitive action by $\operatorname{Aut} C$ with stabilizer G, so $\#F(X) = \#\operatorname{Hom}(C, X) = [\operatorname{Aut} C : G]$ as well; hence $F(\bar{f})$ is injective. We see that any connected object is a finite quotient of a Galois object.

Let $\mathcal{I}$ be the category of pairs (C, c) where $C \in \mathcal{C}$ is Galois and $c \in F(C)$; a morphism $(C, c) \to (D, d)$ is a map $f \colon C \to D$ with $F(f)(c) = d$. If such a map exists, it is unique by $(*)$. For $(C, c), (D, d) \in \mathcal{I}$ there exist a map $f \colon E \to C \times D$ and $e \in F(E)$ with E a Galois object and $F(f)(e) = (c, d)$. By projection we find maps $(E, e) \to (C, c)$ and $(E, e) \to (D, d)$ so $\mathcal{I}$ is cofiltered. We consider the colimit of $\mathcal{I}^{\mathrm{op}} \to \mathrm{FSet}, (C, c) \mapsto \operatorname{Hom}(C, X)$. For $f \colon (C, c) \to (D, d)$ the diagram

$$\begin{tikzcd} \operatorname{Hom}(D, X) \arrow[d, "f^*"'] \arrow[dr] & \\ & F(X) \\ \operatorname{Hom}(C, X) \arrow[ur] & \end{tikzcd}$$

commutes, so we have a natural map $\operatorname{colim}_{(C,c) \in \mathcal{I}^{\mathrm{op}}} \operatorname{Hom}(C, X) \to F(X)$. As the colimit is filtered, this map is injective. Using once more the construction of Galois objects above, we see it is surjective as well. Then functoriality in X says

$$F = \operatorname*{colim}_{(C,c)\in\mathcal{I}^{\mathrm{op}}} \mathrm{Hom}(C,-).$$

We say that F is *prorepresentable* by the *prosystem* $\lim_{(C,c)\in\mathcal{I}} C$. However, be aware that the limit does not necessarily exist in $\mathcal{C}$.

Let $f\colon (C,c) \to (D,d)$ be in $\mathcal{I}$. Since $F(D)$ is a free transitive $\mathrm{Aut}\,D$-set, for $\sigma \in \mathrm{Aut}\,C$ there is a unique $\tau \in \mathrm{Aut}\,D$ with $F(\tau)(d) = F(f\sigma)(c)$ and hence $\tau f = f\sigma$. We get a homomorphism $\mathrm{Aut}\,C \to \mathrm{Aut}\,D$. It is surjective by transitivity of $\mathrm{Aut}\,C$ acting on $\mathrm{Hom}(C,D)$. For $\alpha \in \mathrm{Aut}\,F$ let $\sigma \in \mathrm{Aut}\,C$ be the unique element satisfying $F(\sigma)(c) = \alpha_C(c)$. We obtain a group homomorphism $\mathrm{Aut}\,F \to \mathrm{Aut}\,C$. The diagram

$$\begin{tikzcd} & \mathrm{Aut}\,C \arrow[d, "f_*"] \\ \mathrm{Aut}\,F \arrow[ur] \arrow[r] & \mathrm{Aut}\,D \end{tikzcd}$$

commutes, yielding $\mathrm{Aut}\,F \to \lim_{(C,c)\in\mathcal{I}} \mathrm{Aut}\,C$. Any $\alpha \in \mathrm{Aut}\,F$ is determined by its action on $F(C)$ for C Galois. Therefore $\mathrm{Aut}\,F$ is a closed subgroup of $\prod_{(C,c)\in\mathcal{I}} \mathrm{S}(F(C))$. The compatibility condition to be in $\mathrm{Aut}\,F$ coincides with that to be in $\lim_{(C,c)\in\mathcal{I}} \mathrm{Aut}\,C$, hence

$$\mathrm{Aut}\,F = \lim_{(C,c)\in\mathcal{I}} \mathrm{Aut}\,C$$

as profinite groups. Note that all projections $\mathrm{Aut}\,F \to \mathrm{Aut}\,C$ are surjective because the limit is cofiltered with surjective transition maps.

Since both F and the forgetful functor $\mathrm{Aut}\,F$-$\mathrm{FSet} \to \mathrm{FSet}$ are exact and conservative, the same holds for $\mathcal{F}$. Furthermore $\mathcal{F}$ preserves connectedness. Indeed, let $X \in \mathcal{C}$ be connected. Write $X = C/G$ for some Galois C and $G \subseteq \mathrm{Aut}\,C$. Then we have $\mathcal{F}(X) = \mathcal{F}(C)/G = \mathrm{Aut}\,C/G$. Since $\mathrm{Aut}\,F \to \mathrm{Aut}\,C$ is surjective, $\mathrm{Aut}\,C/G$ is a transitive $\mathrm{Aut}\,F$-set, so $\mathcal{F}(X)$ is connected.

At last we get to the theorem statement. We show that $\mathcal{F}$ is essentially surjective, i.e. that any finite $\mathrm{Aut}\,F$-set P is of the form $\mathcal{F}(X)$ for some $X \in \mathcal{C}$. We may assume P is transitive. Then $P \cong \mathrm{Aut}\,C/G$ for some Galois object C and $G \subseteq \mathrm{Aut}\,C$. As before we have $P \cong \mathcal{F}(C/G)$.

We prove $\mathcal{F}$ is fully faithful, i.e. for $X, Y \in \mathcal{C}$ the map $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(\mathcal{F}(X), \mathcal{F}(Y))$ is a bijection. It is certainly injective by reflection of equalizers. So it suffices to show both sides are equipotent. Since $\mathcal{F}$ preserves connected components, one may reduce to the case where X and Y are connected. Write $X = C/G$ and $Y = C/H$ for some $(C,c) \in \mathcal{I}$ and $G, H \subseteq \mathrm{Aut}\,C$. For $f\colon X \to Y$ there exists $\sigma \in \mathrm{Aut}\,C$ such that $[F(\sigma)(c)] = F(f)([c])$ in $F(C)/H$, and then $f = \bar{\sigma}$. The coset $H\sigma$ is well-defined and $\sigma \in \mathrm{Aut}\,C$ descends to a map $X \to Y$ if and only if $G \subseteq \sigma^{-1}H\sigma$. Thus $\#\mathrm{Hom}(X,Y) = \#\{H\sigma\colon G \subseteq \sigma^{-1}H\sigma\}$. That coincides with the number of $\mathrm{Aut}\,C$-maps $\mathrm{Aut}\,C/G \to \mathrm{Aut}\,C/H$ so we are done. ■

The Galois correspondence does not really depend on the fundamental functor. This is analogous to the fact that the fundamental group of a topological space does not really depend on the choice of base point, and that the Galois group of a field does not really depend on the choice of a separable closure.

Theorem 2.5. *Let $\mathcal{C}$ be a category and $F, F' : \mathcal{C} \to \mathrm{FSet}$ two functors such that $(\mathcal{C}, F)$ and $(\mathcal{C}, F')$ are Galois categories. Then $F \cong F'$ and in particular $\operatorname{Aut} F \cong \operatorname{Aut} F'$.*

Consequently, if π-FSet is equivalent to π'-FSet then $\pi \cong \pi'$.

Proof. Let $\mathcal{I}$ be as before and $\mathcal{I}'$ the same for F'. For each Galois object $C \in \mathcal{C}$ choose one element $c \in F(C)$ and one element $c' \in F'(C)$. Let $\mathcal{J} \subseteq \mathcal{I}$ and $\mathcal{J}' \subseteq \mathcal{I}'$ be the corresponding full subcategories. We consider $f\colon (C, c) \to (D, d)$ in $\mathcal{J}$ with corresponding objects $(C, c'), (D, d')$ in $\mathcal{J}'$. There is an automorphism $\tau \in \operatorname{Aut} D$ satisfying $F'(\tau f)(c') = d'$, yielding a morphism $\tau f\colon (C, c') \to (D, d')$ in $\mathcal{J}'$. So morphisms f in $\mathcal{J}$ correspond to morphisms f' in $\mathcal{J}'$.

For $\sigma \in \operatorname{Aut} C$ there is a unique $\tau \in \operatorname{Aut} D$ with $f'\sigma = \tau f$. This map $\operatorname{Aut} C \to \operatorname{Aut} D$ is surjective by transitivity of $\operatorname{Aut} C$ acting on $\operatorname{Hom}(C, D)$. Hence the cofiltered limit $\lim_{(C,c)\in\mathcal{J}} \operatorname{Aut} C$ is non-empty and there exists a system $(\alpha_C)_C \in \prod_{(C,c)\in\mathcal{I}} \operatorname{Aut} C$ satisfying $f'\alpha_C = \alpha_D f$ for all $f\colon (C, c) \to (D, d)$ in $\mathcal{J}$ corresponding to f' in $\mathcal{J}'$. Then we have

$$F = \operatorname*{colim}_{(C,c)\in\mathcal{J}^{\mathrm{op}}} \operatorname{Hom}(C, -) \cong \operatorname*{colim}_{(C,c')\in\mathcal{J}'^{\mathrm{op}}} \operatorname{Hom}(C, -) = F'$$

where the middle isomorphism is induced by α and the outer identifications hold because $\mathcal{J} \subseteq \mathcal{I}$ and $\mathcal{J}' \subseteq \mathcal{I}'$ are initial subcategories. ■

Theorem 2.6. *Let $(\mathcal{C}, F)$ and $(\mathcal{C}', F')$ be Galois categories and let $G\colon \mathcal{C} \to \mathcal{C}'$ be such that $F = F'G$. There is a natural map $g\colon \operatorname{Aut} F' \to \operatorname{Aut} F$ such that*

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\quad G \quad} & \mathcal{C}' \\
{\scriptstyle \mathcal{F}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{F}'} \\
\operatorname{Aut} F\text{-FSet} & \xrightarrow{g^*} & \operatorname{Aut} F'\text{-FSet}
\end{array}$$

commutes up to 2-isomorphism.

Proof. Define g by sending $\alpha' \in \operatorname{Aut} F'$ to the element $\alpha \in \operatorname{Aut} F$ satisfying $\alpha_X = \alpha'_{G(X)}$ in $\mathrm{S}(F(X)) = \mathrm{S}(F'G(X))$. It is a continuous homomorphism by exercise 2D and has the desired property by construction. ■

Topological coverings

As a first application we prove theorem 1.2. In the newly-developed terminology it says that for any connected topological space X and $x \in X$ the pair $(\operatorname{FCov} X, F_x)$ is a Galois category. Here is the technical ingredient.

Lemma 2.7. *Let $p\colon Y \to X$, $q\colon Z \to X$ be finite coverings and $f\colon Y \to Z$ a covering map. Each $x \in X$ has an open neighborhood U above which Y and Z are trivial and such that*

$$\begin{array}{ccc}
p^{-1}(U) & \xrightarrow{\quad f \quad} & q^{-1}(U) \\
\Big\| & & \Big\| \\
U \times p^{-1}(x) & \xrightarrow{\mathrm{id}_U \times f} & U \times q^{-1}(x)
\end{array}$$

commutes. ♦

Exercise 2G. Prove this. ♦

The empty covering $\varnothing \to X$ and the trivial covering $X \to X$ are initial respectively final objects in FCov X. For covering maps $f\colon Y \to W$, $g\colon Z \to W$ we claim that

$$Y \times_W Z = \{(y,z) \in Y \times Z\colon f(y) = g(z)\}$$

is the fiber product in FCov X. It is the fiber product in Top so it suffices to show that it is a finite covering of X. That question is local on X, hence we can reduce to the situation in the lemma, where it is clear. Similarly one sees that for coverings maps $f\colon W \to Y$, $g\colon W \to Z$ the pushout is given by

$$Y \sqcup_W Z = (Y \sqcup Z)/{\sim}$$

with $\sim$ the equivalence relation generated by $f(w) \sim g(w)$ for $w \in W$. Hence FCov X has finite limits and finite colimits.

Exercise 2H. Using the constructions above, show that F_x is exact. ♦

Let $f\colon Y \to Z$ be a covering map and suppose $F_x(f)$ is bijective. If $F_u(f)$ is bijective for some $u \in X$, then by the lemma $F_v(f)$ is bijective for all v in an open neighborhood of u. Therefore $\{u \in X\colon F_u(f) \text{ bijective}\}$ is open in X. By the same argument its complement is open. As X is connected, $F_x(f)$ being bijective implies that $F_u(f)$ is bijective for all $u \in X$, hence that f is bijective. As coverings are open, f is an isomorphism. So F_x is conservative.

As F_x is exact and conservative, f is a mono- or epimorphism if and only if $F_x(f)$ is. Reasoning as above, $F_x(f)$ is injective or surjective if and only if f is; so the monomorphisms of FCov X are the injections and the epimorphisms the surjections. A last application of lemma 2.7 shows that for any covering map $f\colon Y \to Z$ the image $\operatorname{im} f \subseteq Z$ is open and closed, hence a finite covering of X. Then $Y \to \operatorname{im} f \to Z$ is an epi-mono-factorization. If f itself is a monomorphism, then $Z \setminus \operatorname{im} f$ is also a finite covering of X and acts as a complement. This proves the first part of theorem 1.2.

Exercise 2I. Deduce the second part of theorem 1.2 from theorem 2.5. ♦

Exercise 2J. Let Y be a second connected topological space, $y \in Y$ a point, and $f\colon X \to Y$ a continuous map sending x to y. Show that there is a natural map $f_*\colon \hat{\pi}(X,x) \to \hat{\pi}(Y,y)$. ♦

Exercise 2K. Show that $\hat{\pi}(X,x)$ is trivial if X is irreducible. ♦

3. The étale fundamental group

Transferring the topological example to the algebraic setting, one has to determine the correct notion of 'finite coverings'. These turn out to be the finite étale morphisms. We will show that the category of finite étale schemes over a scheme X, together with a suitable fiber functor, is a Galois category. The associated Galois group will be the étale fundamental group of X.

Finite étale morphisms

Let's begin with a short discussion of (finite) étale morphisms. All schemes are tacitly assumed locally noetherian. That is not necessary but simplifies some technical details.

Definition 3.1. A morphism of schemes $p\colon Y \to X$ is *flat* if for all $y \in Y$ the local ring map $\mathcal{O}_{X,p(y)} \to \mathcal{O}_{Y,y}$ is flat. A morphism of schemes $p\colon Y \to X$ is *unramified* if it is locally of finite type and for all $y \in Y$ the map $\mathcal{O}_{X,p(y)}/\mathfrak{m}_{p(y)} \to \mathcal{O}_{Y,y}/\mathfrak{m}_{p(y)}\mathcal{O}_{Y,y}$ is a finite separable extension of fields. A morphism of schemes is *étale* if it is flat and unramified. ♦

For who knows what it means, étale is equivalent to smooth of relative dimension 0. From the definitions it is clear that being étale is stable under composition, stable under base change, and local on the domain and codomain.

Recall that a scheme map $Y \to X$ is *finite* if for all affine open $U \subseteq X$ the inverse image $V \subseteq Y$ is also affine and $\mathcal{O}_Y(V)$ is a finitely generated $\mathcal{O}_X(U)$-module.

Example 3.2. Let K be a field. A map $Y \to \operatorname{Spec} K$ is étale if and only if it is unramified and if and only if Y is a disjoint union of schemes of the form $\operatorname{Spec} L$ where L is a finite separable field extension of K. It is finite étale if the disjoint union is finite. ♦

Example 3.3. Let A be a ring and $f \in A[t]$ a monic non-constant polynomial. The morphism $\operatorname{Spec} A[t]/(f) \to \operatorname{Spec} A$ is finite. It is étale if and only if the discriminant $\Delta(f)$ is a unit in A.♦

An important property of finite étale morphisms is that they satisfy 'faithfully flat descent'. We omit the proof.

Lemma 3.4. *Let $A \to A'$ be a faithfully flat ring homomorphism. A map $Y \to \operatorname{Spec} A$ is finite, flat, or unramified if and only if the base change $Y \times_A A' \to \operatorname{Spec} A'$ is.* ♦

Let A be a ring and B an A-algebra that is free of finite rank as A-module. The *trace* $\operatorname{Tr}_{B/A}(b)$ of $b \in B$ over A is the trace of the A-linear map $B \to B, c \mapsto cb$.

Proposition 3.5. *A morphism $Y \to X$ is finite étale if and only if every $x \in X$ has an affine open neighborhood $U \subseteq X$ whose inverse image $V \subseteq Y$ is affine as well such that, writing $A := \mathcal{O}_X(U)$ and $B := \mathcal{O}_Y(V)$, the A-module B is free of finite rank and the map*

$$B \to \operatorname{Hom}_A(B, A), \quad b \mapsto \big(c \mapsto \operatorname{Tr}_{B/A}(bc)\big)$$

is an isomorphism of A-modules.

Proof. A morphism $p\colon Y \to X$ is *locally free* if every $x \in X$ has an affine open neighborhood $U \subseteq X$ whose inverse image $V \subseteq Y$ is affine with $\mathcal{O}_Y(V)$ is a free $\mathcal{O}_X(U)$-module. Since free modules are flat, finite locally free morphisms are finite flat. Conversely suppose p is finite flat, take $U \subseteq X$ affine open with inverse image $V \subseteq Y$ and set $A := \mathcal{O}_X(U)$ and $B := \mathcal{O}_Y(V)$. For all $\mathfrak{p} \in \operatorname{Spec} A$ the $A_\mathfrak{p}$-module $B_\mathfrak{p}$ is finitely generated and flat over a local ring, hence free. As X is locally noetherian this implies that p is finite locally free.

It remains to prove that for any ring A and A-algebra B that is free of finite rank as module, B satisfies the stated trace condition if and only if $\operatorname{Spec} B \to \operatorname{Spec} A$ is unramified. First suppose that A is an algebraically closed field. If $\operatorname{Spec} B \to \operatorname{Spec} A$ is unramified, then $B = \prod_{i=1}^n A$ for some $n \in \mathbb{N}$ and the trace condition holds. Conversely, assume the trace condition. If $b \in B$ is nilpotent, multiplication by bc is nilpotent for any $c \in B$ hence $c \mapsto \operatorname{Tr}_{B/A}(bc)$ is zero. Therefore B has no non-zero nilpotents. But any finite-dimensional algebra over a field is a finite product of local rings with nilpotent maximal ideals, so B is a finite product of fields. Each field is finite over the algebraically closed field A so $B = \prod_{i=1}^n A$ and $\operatorname{Spec} B \to \operatorname{Spec} A$ is unramified.

Now suppose A is an arbitrary field. Let A' be an algebraic closure and write $B' := A' \otimes_A B$. The map $t\colon B \to \operatorname{Hom}_A(B, A)$ is an isomorphism if and only if $A' \otimes_A t$ is. Since the square

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\operatorname{Tr}_{B/A}"'] & B' \arrow[d, "\operatorname{Tr}_{B'/A'}"] \\ A \arrow[r] & A' \end{tikzcd}$$

commutes, $A' \otimes_A t$ coincides with the trace map t' for $A' \to B'$. In combination with lemma 3.4 we see that t is an isomorphism if and only if $\operatorname{Spec} B \to \operatorname{Spec} A$ is unramified.

Let A be any ring. The map $\operatorname{Spec} B \to \operatorname{Spec} A$ is unramified if and only if for all $\mathfrak{p} \in \operatorname{Spec} A$ the base change to the field $\kappa(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ is. Also $t\colon B \to \operatorname{Hom}_A(B, A)$ is an isomorphism if and only if all its base changes $\kappa(\mathfrak{p}) \otimes_A t$ are. These base changes are the trace maps for $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}) \otimes_A B$, so we are done by the previous paragraph. ■

Take care: for arbitrary affine open $U \subseteq X$ with inverse image $V \subseteq Y$ the $\mathcal{O}_X(U)$-module $\mathcal{O}_Y(V)$ is finitely generated, but in general not free; instead it is *projective*. We avoid the work required to state a suitable trace condition for projective algebras.

The *degree* of Y at $x \in X$ is $\deg_{Y/X}(x) := \operatorname{rk}_A B$ where A and B are as in the proposition. This is well-defined. The map $\deg_{Y/X}\colon X \to \mathbb{N}$ is locally constant. In particular, for $n \in \mathbb{N}$ the set $\{x \in X\colon \deg_{Y/X}(x) = n\}$ is open and closed.

Definition 3.6. A morphism $Y \to X$ is *trivial* if Y is a disjoint union of copies of X. It is *totally split* if each $x \in X$ has an open neighborhood $U \subseteq X$ above which Y is trivial. ♦

Proposition 3.7. *A morphism $Y \to X$ is finite étale if and only if there exists a finite étale surjection $X' \to X$ such that the base change $Y \times_X X' \to X'$ is finite totally split.* ♦

Proof. If $Y \times_X X' \to X'$ is finite totally split, then it is finite étale. Applying lemma 3.4 to affine opens of X we see that $Y \to X$ is finite étale as well.

Suppose $Y \to X$ is finite étale. Since the sets $\{x \in X\colon \deg_{Y/X}(x) = n\}$ are open and closed we reduce to the case where $\deg_{Y/X}$ has a constant value n. We proceed by induction to n. If $\deg_{Y/X} = 0$ then $Y \to X$ is already finite totally split and we are done. Take $n > 0$. We will show below that the diagonal $Y \to Y \times_X Y$ is an open and closed immersion, so we can write $Y \times_X Y = Y \sqcup Z$. The projections $Y \times_X Y \to Y$ are finite étale of degree n. As id_Y has degree 1 it follows that $Z \to Y$ is finite étale of degree $n - 1$. By induction there exists a finite étale surjection $Y' \to Y$ for which $Z \times_Y Y' \to Y'$ is finite totally split. The composition $Y' \to Y \to X$ is a finite étale surjection with the desired property.

It remains to prove that for finite étale maps $Y \to X$ the diagonal $Y \to Y \times_X Y$ is open and closed. This is a local question so we may suppose that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ where B is a finite rank free A-module satisfying the trace condition in proposition 3.5. Then $C := B \otimes_A B$ is a finite rank free B-module satisfying the same trace condition. Let $e \in C$ be the element that corresponds under $C \to \operatorname{Hom}_B(C, B)$ to the multiplication map $m\colon C \to B, b \otimes c \mapsto bc$. Fix $b \in C$. For all $c \in C$ one computes

$$\operatorname{Tr}_{C/B}(ebc) = m(bc) = m(b)m(c) = m(b)\operatorname{Tr}_{C/B}(ec) = \operatorname{Tr}_{C/B}(m(b)ec)$$

so by the trace condition $eb = m(b)e$. In particular we have $e \ker m = 0$. The diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker m \arrow[r] \arrow[d, "0"] & C \arrow[r, "m"] \arrow[d, "\cdot e"] & B \arrow[r] \arrow[d, "\cdot m(e)"] & 0 \\
0 \arrow[r] & \ker m \arrow[r] & C \arrow[r, "m"] & B \arrow[r] & 0
\end{tikzcd}$$

is commutative with split exact rows hence one has $m(e) = \operatorname{Tr}_{C/B}(e) = m(1) = 1$. The identity $eb = m(b)e$ proves e is an idempotent. Therefore the map $B \oplus \ker m \to C, (b, c) \mapsto be + c$ is a multiplicative isomorphism of B-modules. Because B and C have identity elements, so does $\ker m$ and we find $C = B \times \ker m$ as B-algebras. ■

The fundamental theorem

Let X be a connected locally noetherian scheme and denote by $\operatorname{FEt} X$ the category of finite étale schemes over X. Let $\bar{x}$ be a geometric point of X, i.e. a map $\operatorname{Spec} \bar{K} \to X$ where $\bar{K}$ is a separably closed field. We define

$$F_{\bar{x}} \colon \operatorname{FEt} X \to \mathrm{FSet}, \quad Y \mapsto Y \times_X \bar{x}$$

where we note that $Y \times_X \bar{x}$ is a finite étale $\bar{K}$-scheme, hence just a finite set.

Theorem 3.8. *The pair* $(\operatorname{FEt} X, F_{\bar{x}})$ *is a Galois category.* ♦

Definition 3.9. The *étale* or *algebraic fundamental group* of $(X, \bar{x})$ is $\pi_{\mathrm{et}}(X, \bar{x}) := \operatorname{Aut} F_{\bar{x}}$. ♦

The following two exercises explain how theorem 3.8 generalizes Galois theory.

Exercise 3A. Show that Spec is an anti-equivalence $\operatorname{FSep} K \to \operatorname{FEt} K$. Give a geometric description of the finite étale K-schemes that correspond to finite separable field extensions. ♦

Exercise 3B. Show that theorem 1.3 is the special case $X = \operatorname{Spec} K$ of theorem 3.8. ♦

The proof that $(\operatorname{FEt} X, F_{\bar{x}})$ is a Galois category mirrors closely the proof for topological spaces. Here is the algebraic analogue of lemma 2.7.

Lemma 3.10. *Let $Y \to X, Z \to X$ be finite totally split and $f \colon Y \to Z$ a covering map. Each $x \in X$ has an open neighborhood U above which Y and Z are trivial and such that*

$$\begin{tikzcd} Y_U \arrow[r, "f"] \arrow[d, equal] & Z_U \arrow[d, equal] \\ U \times Y_x \arrow[r, "\mathrm{id}_U \times f"] & U \times Z_x \end{tikzcd}$$

commutes. ♦

Exercise 3C. Prove this. ♦

Exercise 3D. Show that if $Y \to X$ and $Z \to X$ are finite étale and $f \colon Y \to Z$ is a covering map, then f is finite étale as well. ♦

The empty covering $\varnothing \to X$ and the trivial covering $X \to X$ are initial respectively final objects in $\operatorname{FEt} X$. For covering maps $f \colon Y \to W$ and $g \colon Z \to W$ also $Y \times_W Z$ is finite étale over X so it is the fiber product in $\operatorname{FEt} X$.

Constructing pushouts is more complicated. Let $f \colon W \to Y, g \colon W \to Z$ be covering maps. We construct the pushout $Y \sqcup_W Z$ in the case X is affine; then it exists for general X by gluing. Write $A := \mathcal{O}_X(X)$, $B := \mathcal{O}_Y(Y)$, $C := \mathcal{O}_Z(Z)$, and $D := \mathcal{O}_W(W)$. The ring

$$B \times_D C := \{(b, c) \in B \times C \colon f^\#(b) = g^\#(c)\}$$

is the fiber product of $f^\#$ and $g^\#$ in A-Alg. By duality $\operatorname{Spec}(B \times_D C)$ is the pushout of f and g in the category of affine schemes over X. We have to show that it is finite étale.

There exists a finite étale surjection $X' \to X$ such that, with obvious notation, Y', Z' and W' are totally split over X'. If Y', Z' and W' are trivial over X' and f', g' are as in lemma 3.10, then $\operatorname{Spec}(B' \times_{D'} C')$ is trivial over X' as well. But locally on X' this is the case, so in general $\operatorname{Spec}(B' \times_{D'} C')$ is finite totally split over X'. Since one has $B' \times_{D'} C' = A' \otimes_A (B \times_D C)$ it follows that $\operatorname{Spec}(B \times_D C)$ is finite étale over X. We conclude that $\operatorname{FEt} X$ has finite limits and finite colimits.

Exercise 3E. Show that $F_{\bar{x}}$ is exact. ⧫

Next we classify the mono- and epimorphisms of $\mathrm{FEt}\,X$. Take a covering map $Y \to Z$. Let $U \subseteq X$ be affine open with inverse images $V \subseteq Y, W \subseteq Z$ and write $A := \mathcal{O}_X(U)$, $B := \mathcal{O}_Y(V)$ and $C := \mathcal{O}_Z(W)$. If $Y \to Z$ is a monomorphism, the diagonal $Y \to Y \times_Z Y$ is an isomorphism, hence $B \otimes_C B \cong B$. By local rank considerations this implies that for any $z \in Z$ the degree $\deg_{Y/Z}(z)$ is either 0 or 1. We find $Z = Z_0 \sqcup Y$ with $Z_0 = \{z \in Z\colon \deg_{Y/Z}(z) = 0\}$, so $Y \to Z$ is an open and closed immersion. Conversely any open and closed immersion is a monomorphism. Moreover, we see that subobjects in $\mathrm{FEt}\,X$ admit complements.

For any covering map $Y \to Z$ we have $Z = Z_0 \sqcup Z_1$ with $Z_0 := \{z \in Z\colon \deg_{Y/Z}(z) = 0\}$ and $Z_1 := \{z \in Z\colon \deg_{Y/Z}(z) > 0\}$. The two morphisms $Z_0 \sqcup Z_1 \rightrightarrows Z_0 \sqcup Z_0 \sqcup Z_1$ coincide after precomposition with $Y \to Z$. If $Y \to Z$ is an epimorphism, they must coincide themselves. This is true only if $Z_0 = \varnothing$, i.e. if $Y \to Z$ is surjective. Conversely assume $Y \to Z$ is surjective. Restrict to the affine setting with notation as before. For each $\mathfrak{p} \in \operatorname{Spec} C$ the $C_\mathfrak{p}$-module $B_\mathfrak{p}$ is free of rank $\deg_{B/C}(\mathfrak{p}) \geq 1$ by surjectivity of $Y \to Z$, so $C_\mathfrak{p} \to B_\mathfrak{p}$ is injective. It follows that $C \to B$ is injective and that the diagonal $C \to C \times_B C$ is an isomorphism. Globally we find that the codiagonal $Z \sqcup_Y Z \to Z$ is an isomorphism hence $Y \to Z$ is an epimorphism.

Exercise 3F. Show that each map in $\mathrm{FEt}\,X$ has factors into an epi- and a monomorphism. ⧫

Exercise 3G. Let $Y \to X, Z \to X$ be finite étale morphisms. Show that a covering map $Y \to Z$ is a monomorphism in $\mathrm{FEt}\,X$ if and only if $\deg_{Y/Z}(z) \leq 1$ for all $z \in Z$, and an epimorphism if and only if $\deg_{Y/Z}(z) \geq 1$ for all $z \in Z$. ⧫

Exercise 3H. Show that a morphism in $\mathrm{FEt}\,X$ is an isomorphism if and only if it is both a monomorphism and an epimorphism. ⧫

It remains to prove that the fiber functor $F_{\bar{x}}$ is conservative. Let $f\colon Y \to Z$ be a covering map and factor $f = hg$ with $g\colon Y \to W$ an epimorphism and $h\colon W \to Z$ a monomorphism. Since $F_{\bar{x}}$ is exact, $F_{\bar{x}}(g)$ is surjective and $F_{\bar{x}}(h)$ and is injective. So if $F_{\bar{x}}(f)$ is a bijection, both $F_{\bar{x}}(g)$ and $F_{\bar{x}}(h)$ are bijections. Writing $Z = Z_0 \sqcup W$ we get $\deg_{Z_0/X}(x) = \#F_{\bar{x}}(Z_0) = 0$. By connectedness of X the degree of $Z_0 \to X$ is 0 everywhere so $Z_0 = \varnothing$ and h is an isomorphism. On the other hand, choose a finite étale surjection $X' \to X$ such that $Y' := Y \times_X X'$ and $W' := W \times_X X'$ are totally split over X'. Let $\bar{x}'$ be a geometric point of X' lying over $\bar{x}$. Then $F_{\bar{x}'}(g')$ is still a bijection. As in the topological case, lemma 3.10 shows that $F_{\bar{u}'}(g')$ is a bijection for all geometric points $\bar{u}'$ of X' and therefore that g' is an isomorphism. Since degree is stable under base change, g is finite étale of constant degree 1, hence an isomorphism as well. The same follows for f. This concludes the proof of theorem 3.8.

Exercise 3I. Let Y be a second connected locally noetherian scheme, $\bar{y}\colon \operatorname{Spec} \bar{L} \to Y$ a geometric point, and $f\colon X \to Y, \bar{f}\colon \operatorname{Spec} \bar{K} \to \operatorname{Spec} \bar{L}$ morphisms such that the diagram

$$\begin{tikzcd} \operatorname{Spec} \bar{K} \arrow[r, "\bar{f}"] \arrow[d, "\bar{x}"'] & \operatorname{Spec} \bar{L} \arrow[d, "\bar{y}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes. Show that there is a natural map $f_*\colon \pi_{\mathrm{et}}(X, \bar{x}) \to \pi_{\mathrm{et}}(Y, \bar{y})$. ⧫

Exercise 3J. Let L/K be a field extension, $\bar{L}$ a separable closure of L, and $\bar{K}$ the separable closure of K in $\bar{L}$. Show that there is a natural map $\mathrm{Gal}(\bar{L}/L) \to \mathrm{Gal}(\bar{K}/K)$ and that it is given by restricting the action of $\sigma \in \mathrm{Gal}(\bar{L}/L)$ to $\bar{K}$. ⧫

References

[1] A. Grothendieck, M. Raynaud, *Revêtements étales et groupe fondamental* (SGA 1). Lecture Notes in Mathematics 224, Springer, 1971.
Updated edition: `http://arxiv.org/abs/math/0206203`, 2002.

[2] H.W. Lenstra, *Galois theory for schemes*.
`http://websites.math.leidenuniv.nl/algebra/GSchemes.pdf`, 2008.

[3] T. Szamuely, *Galois groups and fundamental groups*.
Cambridge Studies in Advanced Mathematics 117, Cambridge University Press, 2009.