The pro-étale fundamental group

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1. Infinite Galois theory

We develop an 'infinite' version of Grothendieck's Galois theory. It was introduced first by Noohi [3] and slightly modified by Bhatt–Scholze [2].

Definition 1.1. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. A *subobject* of X is a monomorphism $Y \to X$. We say that X is *connected* if it has precisely two isomorphism classes of subobjects. ◆

Definition 1.2. Let $\mathcal{C}$ be a category and $F\colon \mathcal{C} \to \mathrm{Set}$ a functor. Then $\mathcal{C}$ is an *infinite Galois category* with *fundamental functor* F if

- $\mathcal{C}$ has colimits and finite limits,
- F preserves colimits and finite limits,
- F is faithful and conservative,
- each $X \in \mathcal{C}$ is a coproduct of connected objects, and
- $\operatorname{Aut} F \curvearrowright F(X)$ is transitive for each connected $X \in \mathcal{C}$.

For technical reasons we should also assume that the subcategory of connected objects in $\mathcal{C}$ is essentially small; however, we ignore that. ◆

Remark 1.3. This is what Bhatt–Scholze call a *tame* infinite Galois category. ◆

Example 1.4. Let G be a topological group. Then G-Set with the forgetful functor $G\text{-Set} \to \mathrm{Set}$ is an infinite Galois category. ◆

Let $(\mathcal{C}, F)$ be an infinite Galois category. Endow $\operatorname{Aut} F \subseteq \prod_{X \in \mathcal{C}} \mathrm{S}(F(X))$ with the induced topology, where each $\mathrm{S}(F(X))$ is given the compact-open topology. (If $F(X)$ is finite, this coincides with the discrete topology.) Then $\operatorname{Aut} F$ acts continuously on each $F(X)$, so we get a functor $\mathcal{F}\colon \mathcal{C} \to \operatorname{Aut} F\text{-Set}$.

Theorem 1.5. *$\mathcal{F}$ is an equivalence.*

Proof. Take $X \in \mathcal{C}$, and write $X = \coprod_{i \in I} X_i$ with each X_i connected. Then $F(X) = \coprod_{i \in I} F(X_i)$ and $\operatorname{Aut} F$ acts transitively on each $F(X_i)$. In other words, $\mathcal{F}$ preserves connected components. Hence, the subobjects of X correspond bijectively to the subobjects of $\mathcal{F}(X)$. Identifying a map $f\colon X \to Y$ with its graph $\Gamma_f\colon X \to X \times Y$, we see

$$\begin{aligned} \operatorname{Hom}(X, Y) &= \{\text{subobjects } \Gamma\colon X \to X \times Y \text{ with } \pi_X \circ \Gamma = \mathrm{id}_X\} \\ &= \{\text{subobjects } \Delta\colon \mathcal{F}(X) \to \mathcal{F}(X) \times \mathcal{F}(Y) \text{ with } \pi_{\mathcal{F}(X)} \circ \Delta = \mathrm{id}_{\mathcal{F}(X)}\} \\ &= \operatorname{Hom}(\mathcal{F}(X), \mathcal{F}(Y)) \end{aligned}$$

using that F is faithful and conservative. We conclude that $\mathcal{F}$ is fully faithful.

Let $U \subseteq \operatorname{Aut} F$ be an open subgroup; we show that $\operatorname{Aut}(F)/U$ is in the essential image of $\mathcal{F}$. The topology shows that there are $X_1, \ldots, X_n \in \mathcal{C}$ and $x_1 \in \mathcal{F}(X_1), \ldots, x_n \in \mathcal{F}(X_n)$ such that U contains the simultaneous stabilizer $V \subseteq \operatorname{Aut} F$ of $x_1, \ldots, x_n$. Let Y be the connected component of $X_1 \times \ldots \times X_n$ with $(x_1, \ldots, x_n) \in F(Y)$. Then $\mathcal{F}(Y) \cong \operatorname{Aut}(F)/V$ as $\operatorname{Aut} F$-sets. Being a colimit, the quotient $X = Y/(U/V)$ exists, and $\mathcal{F}(X) \cong \operatorname{Aut}(F)/U$. ■

2. Noohi groups

We shall now classify the type of topological group that arises as $\operatorname{Aut} F$. First recall the classical situation: let G be a topological group and $F\colon G\text{-FSet} \to \text{FSet}$ the forgetful functor, then G is profinite if and only if the natural map $G \to \operatorname{Aut} F$ is an isomorphism.

Definition 2.1. Let G be a topological group and $F\colon G\text{-Set} \to \text{Set}$ the forgetful functor. Then G is *Noohi* if the natural map $G \to \operatorname{Aut} F$ is an isomorphism. ◆

Example 2.2. Let X be a set. Then $\mathrm{S}(X)$ with the compact-open topology is Noohi. To see this, note that the canonical $\mathrm{S}(X)$-action on X is continuous. We get a projection $\operatorname{Aut} F \to \mathrm{S}(X)$, and the composition $\mathrm{S}(X) \to \operatorname{Aut} F \to \mathrm{S}(X)$ is the identity. So certainly $\mathrm{S}(X) \to \operatorname{Aut} F$ is injective.

Let $U \subseteq \mathrm{S}(X)$ be an open subgroup. There exists a finite subset $F \subseteq X$ whose pointwise stabilizer S_F is contained in U. (These form a basis of open neighborhoods of the identity.) The natural map $\mathrm{S}(X) \to X^F$ yields an $\mathrm{S}(X)$-equivariant injection $\mathrm{S}(X)/S_F \to X^F$. Now choose $\alpha \in \operatorname{Aut} F$. Its action on X determines its action on X^F, hence on $\mathrm{S}(X)/S_F$, hence on $\mathrm{S}(X)/U$. So α is already determined by its action on X, i.e. $\mathrm{S}(X) \to \operatorname{Aut} F$ is surjective. ◆

In the classical case, there is also a fully topological characterization: a topological group G is profinite if and only if G is totally disconnected, compact and Hausdorff. For Noohi groups, we have the following.

Theorem 2.3 ([2], 7.1.5). *A topological group G is Noohi if and only if G is complete and its open subgroups form a basis of open neighborhoods of $1 \in G$.* ◆

Here *complete* means that G is Hausdorff and closed in all its topological supergroups. Equivalently, G is Raĭkov complete, or complete for its two-sided uniformity. Details can be found in [1], §3.6; we just remark that any Hausdorff group G has a natural completion G^*, and $G \subseteq G^*$ is dense.

Lemma 2.4. *Let G be a Hausdorff group and $U \subseteq G$ an open subgroup. If U is Noohi, then so is G.*

Proof. If the open subgroups of U form a basis of open neighborhoods of $1 \in U$, they also do so in G. It remains to prove that G is complete. Let G^* be its completion. Being Noohi, U is closed in G. By assumption $U \subseteq G$ is also open, hence $G = \coprod_{g \in G/U} gU$. Taking closures in G^*, we get

$$G^* = \bar{G} = \coprod_{g \in G/U} \overline{gU} = \coprod_{g \in G/U} gU = G,$$

using again that U is Noohi, hence closed in G^*. ■

Locally compact Hausdorff groups are complete. This yields lots of examples of Noohi groups: discrete groups, profinite groups, local fields, rings of integers in local fields. In another direction, $\overline{\mathbb{Z}}_\ell$ (endowed with the colimit topology) is Noohi. Indeed, since $\overline{\mathbb{Z}}_\ell$ is abelian, we have $\operatorname{Aut} F = \lim_U \overline{\mathbb{Z}}_\ell/U$, where the limit is taken over all open subgroups; and the natural map $\overline{\mathbb{Z}}_\ell \to \lim_U \overline{\mathbb{Z}}_\ell/U$ is an isomorphism. By the preceding lemma, also $\overline{\mathbb{Q}}_\ell$ is Noohi.

Theorem 2.5. *A topological group G is Noohi if and only if G is isomorphic to* $\operatorname{Aut} F$ *for some infinite Galois category $(\mathcal{C}, F)$.*

Proof. If G is Noohi, then G-Set with the forgetful functor $F\colon G\text{-Set} \to \text{Set}$ is an infinite Galois category, and by definition $G \cong \operatorname{Aut} F$. Conversely, let $(\mathcal{C}, F)$ be an infinite Galois category. Recall that $\operatorname{Aut} F$ is a closed subgroup of $\prod_{X \in \mathcal{C}} \mathrm{S}(F(X))$. By theorem 2.3 being Noohi is stable under taking products and closed subgroups, hence $\operatorname{Aut} F$ is Noohi. ■

3. The pro-étale fundamental group

Let X be a locally noetherian scheme.

Theorem 3.1 ([2], 7.3.9). *For a sheaf $\mathcal{F}$ on $X_{\text{pro-et}}$, the following are equivalent:*

- *$\mathcal{F}$ is* locally constant, *i.e. there is a pro-étale cover $\{U_i \to X\}_{i \in I}$ with each $\mathcal{F}_{U_i}$ constant, and*
- *$\mathcal{F}$ is a* geometric cover, *i.e. $\mathcal{F}$ is representable by an étale X-scheme that satisfies the valuative criterion of properness.* ♦

Recall that $Y \to X$ satisfies the *valuative criterion of properness* if for all discrete valuation rings R with fraction field K and all solid commutative diagrams

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & Y \\ \big\downarrow & \overset{\exists!}{\nearrow} & \big\downarrow \\ \operatorname{Spec} R & \longrightarrow & X \end{array}$$

there is a unique lift $\operatorname{Spec} R \to Y$ making the full diagram commute. If $Y \to X$ were also of finite type, then the valuative criterion is equivalent to properness. But we do not assume any finiteness conditions!

Example 3.2. If $X = \operatorname{Spec} k$ for a field k, then it is easily verified that both the locally constant sheaves and the geometric covers are precisely the sheaves $\mathcal{F} \in \operatorname{Sh}(X_{\text{pro-et}})$ that are pullbacks from $\operatorname{Sh}(X_{\text{et}})$ via the morphism of sites $X_{\text{pro-et}} \to X_{\text{et}}$. ♦

Proof (locally constant $\Rightarrow$ geometric cover). First suppose $\mathcal{F}$ is constant. Then certainly $\mathcal{F}$ is representable by an étale X-scheme that satisfies the valuative criterion of properness; moreover $\mathcal{F}$ is separated. Now if $\mathcal{F}$ is locally constant, by fpqc descent $\mathcal{F}$ is at least an étale separated algebraic space over X satisfying the valuative criterion of properness. But algebraic spaces locally quasi-finite separated over a scheme are representable by schemes. ■

We write $\operatorname{Cov} X \subseteq \operatorname{Sh}(X_{\text{pro-et}})$ for the full subcategory of geometric covers (equivalently, of locally constant sheaves).

Now assume X is connected. Choose a geometric point $\bar{x}$ of X, and let $F_{\bar{x}}\colon \operatorname{Cov} X \to \operatorname{Set}$ be the fibre functor $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$. We define (finally) the pro-étale fundamental group.

Definition 3.3. The *pro-étale fundamental group* of $(X, \bar{x})$ is $\pi^{\text{pro-et}}(X, \bar{x}) = \operatorname{Aut} F_{\bar{x}}$. ♦

Theorem 3.4. *The pair $(\operatorname{Cov} X, F_{\bar{x}})$ is an infinite Galois category.*

Proof. We omit the verifications that $\operatorname{Cov} X$ has colimits and finite limits, that F commutes with them, and that F is faithful and conservative.

Let Y/X be a geometric cover. Since Y is locally noetherian, its connected components are open and closed. The components are geometric covers as well. Suppose Y is connected; we prove that it is a connected object. Let Z be a subobject. The image of Z in Y is open and stable under specializations since $Z \to Y$ is also a geometric cover. Using again that Y is locally noetherian, this implies that the image of Z in Y is closed. By connectedness of Y, the image is either $\varnothing$ or Y. In the latter case $Z \to Y$ is geometric cover and a homeomorphism, hence an isomorphism.

We now show that, for Y connected, the action of $\operatorname{Aut} F_{\bar{x}}$ on $Y_{\bar{x}}$ is transitive. Let $\bar{y}, \bar{y}'$ be lifts of $\bar{x}$ to geometric points of Y. As Y is locally noetherian and connected, there exists a 'path' $\bar{y} = \bar{y}_0, \bar{y}_1, \dots, \bar{y}_n = \bar{y}'$ of specializations and generalizations. Let $\bar{x} = \bar{x}_0, \bar{x}_1, \dots, \bar{x}_n = \bar{x}$ be the images in X. Choose discrete valuation rings R_i and maps $\operatorname{Spec} R_i \to Y$ with $\bar{y}_{i-1}, \bar{y}_i$ as special respectively generic fibre or conversely. By the valuative criterion of properness we obtain isomorphisms of fibre functors

$$F_{\bar{x}} = F_{\bar{x}_0} \cong F_{\bar{x}_1} \cong \dots \cong F_{\bar{x}_n} = F_{\bar{x}}$$

hence an automorphism $\alpha \in \operatorname{Aut} F_{\bar{x}}$. By construction α maps $\bar{y}$ to $\bar{y}'$. ■

We mention some good properties.

Lemma 3.5 ([2], 7.4.3). *The profinite completion $\hat{\pi}^{\text{pro-et}}(X, \bar{x})$ is $\pi^{\text{et}}(X, \bar{x})$.* ♦

Lemma 3.6 ([2], 7.4.10). *If X is geometrically unibranch, then $\pi^{\text{pro-et}}(X, \bar{x}) = \pi^{\text{et}}(X, \bar{x})$.* ♦

Classically, representations of $\pi^{\text{et}}(X, \bar{x})$ contain useful information. For instance, the category of finite rank locally free $\mathbb{Z}_\ell$-sheaves on X is equivalent to the category of continuous representations of $\pi^{\text{et}}(X, \bar{x})$ in finite rank free $\mathbb{Z}_\ell$-modules. However, the analogue for $\mathbb{Q}_\ell$ fails in general.

Theorem 3.7 ([2], 7.4.7). *The category of finite rank locally free $\mathbb{Q}_\ell$-sheaves on X is equivalent to the category of continuous representations of $\pi^{\text{pro-et}}(X, \bar{x})$ in finite dimensional $\mathbb{Q}_\ell$-vector spaces.* ♦

The theorem is actually true for any algebraic extension $K/\mathbb{Q}_\ell$, since $\bar{\mathbb{Q}}_\ell$ is a Noohi group.

4. Example: the nodal curve

Let X be the nodal curve, i.e. the projective line with 0 and ∞ glued together transversally. We have seen its finite étale covers. There is a unique connected degree n finite étale cover $Y_n \to X$; it consists of n copies of $\mathbb{P}^1$ where 0 in the i^{th} copy is identified with ∞ in the $(i+1)^{\text{st}}$ copy, cyclically. We concluded $\pi^{\text{et}}(X, \bar{x}) = \hat{\mathbb{Z}}$.

There is also an étale cover $Y_\infty \to X$, consisting of countably many copies of $\mathbb{P}^1$ glued as before. It is not finite étale, but it is an geometric cover.

Lemma 4.1. *Y_∞ and Y_n, $n \geq 1$ are the only connected geometric covers of the nodal curve X.*

Proof. Let $Y \to X$ be a connected geometric cover. Construct a cartesian diagram

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & \mathbb{P}^1 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

All geometric covers of $\mathbb{P}^1$ are trivial, so $\tilde{Y} = \coprod_{i \in I} \mathbb{P}^1$. Let $U \subset X$ be the complement of the node, and V its inverse image in Y. Then we get a cartesian diagram

$$\begin{array}{ccc} \coprod_{i \in I} \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow \\ V & \longrightarrow & U. \end{array}$$

But $\mathbb{G}_m \to U$ is an isomorphism, so $V = \coprod_{i \in I} \mathbb{G}_m$. Choose $i \in I$ and consider the point 0 in the i^{th} copy of $\mathbb{P}^1$. It is mapped to the node of X, so is identified in Y with precisely one point ∞ of some $\mathbb{P}^1$ (possibly after application of an automorphism of $\mathbb{P}^1$ switching 0 and ∞). Continue this process. If after n steps we get back at the original $\mathbb{P}^1$, then $Y = Y_n$. Otherwise, we have $Y = Y_\infty$. ■

Corollary 4.2. $\pi^{\text{pro-et}}(X, \bar{x}) = \mathbb{Z}$.

Proof. Each geometric cover is a disjoint union of quotients of Y_∞. Therefore $\pi^{\text{pro-et}}(X, \bar{x})$ consists of those permutations of $F_{\bar{x}}(Y) = Y_{\bar{x}} = \mathbb{Z}$ that commute with all automorphisms of Y/X. These automorphisms induce translations on $\mathbb{Z}$, and the only permutations commuting with all translations are the translations themselves. ■

Remark 4.3. Intuitively, $Y_\infty \to X$ is the 'universal' cover. We can now make this precise. Let $(X, \bar{x})$ be a geometrically pointed connected locally noetherian scheme. Suppose $\pi^{\text{pro-et}}(X, \bar{x})$ is discrete. Then the *universal cover* of X is the geometric cover that corresponds under the equivalence $\operatorname{Cov} X \to \pi^{\text{pro-et}}(X, \bar{x})$-Set to the set $\pi^{\text{pro-et}}(X, \bar{x})$ with the regular action. The automorphism group of the universal cover is clearly isomorphic to $\pi^{\text{pro-et}}(X, \bar{x})$ ♦

References

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